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ON THE DISTRIBUTION OF THE MAXIMUM OF A SEMI-MARKOV PROCESS[1]

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0. Introduction. If $\{X(t):t \geqq 0\}$ is a separable stochastic process, the problem of computing the distribution of $Z(t) = \sup [X(s):0 \leqq s \leqq t]$ is of great interest particularly in level crossing (detection) problems and in queuing theory.

Spitzer [8] used combinatorial methods to find the distribution of $Z(t)$ in the case of a discrete time random walk. In [1] Baxter used operator theoretic techniques to give a characterization of the distribution of $Z(t)$ and many other functionals on a discrete time Markov process. In the case of continuous time processes with stationary independent increments Baxter and Donsker [2] obtained the double Laplace transform of the distribution of $Z(t)$. Using a generalization of the classical ballot theorem, Takacs [9], has computed the distribution of $Z(t)$ for many interesting cases involving processes with interchangeable increments.

However, there are many cases in which one must deal with continuous time Markov processes and semi-Markov processes. The purpose of this paper is to extend the results of Baxter [1] by characterizing the distribution of $Z(t)$ for a wide class of semi-Markov processes.

Define $m_{ij}(s)$ to be the Laplace transform of the function $M_{ij}(t) = P[Z(t) = j \mid X_0 = i]$ and let $m(s) = (m_{ij}(s))$. The main result of this paper is in the form of a recurrence relation for $m(s)$

$$m(s) = g(s) + (q(s)m(s))^{\sigma}$$

where $q(s)$ and $g(s)$ are matrices whose elements are Laplace transforms of distributions which occur in the definition of the semi-Markov process and σ is an operator on matrices. Moreover, $m(s)$ is the unique solution of the above equation under a condition on the matrices $q(s)$ which guarantees that the process makes a finite number of transitions in any finite interval of time.

1. Preliminaries. First it is necessary to discuss linear operations defined on a space, $\mathfrak{L}$, of bounded sequences, $\{s_i\}i \varepsilon I$, where I may be an arbitrary subset of the integers. The exact nature of $\mathfrak{L}$ will depend on the state space of the semi-Markov process in question. For us, the important properties of $\mathfrak{L}$ are that it is a Banach space under the supremum norm and that any bounded linear oper-

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ator, A, on $\mathcal{L}$ into $\mathcal{L}$ is of the form

$$A\{s_i\} = \{\textstyle\sum_j a_{ij}s_j\}, \tag{1.1}$$

where $\sum_j |a_{ij}|$ is uniformly bounded in i. Clearly we may identify A with the matrix (a_{ij}), and the norm of $A = \|A\| = \sup_i (\sum_j |a_{ij}|)$.

DEFINITION 1.1. For any operator A of the form (1.1) we define an operator A^σ by

$$\begin{aligned} A^\sigma = (a^\sigma_{ij}) \quad \text{where} \quad a^\sigma_{ij} &= a_{ij} && \text{if} \quad j > i, \\ &= \textstyle\sum_{k \leqq i} a_{i,k} && \text{if} \quad j = i, \\ &= 0 && \text{if} \quad j < i. \end{aligned}$$

We also define $A^\tau = A - A^\sigma$.

Let I be the identity operator, then the following properties hold.

(1.2)
(i) $I^\sigma = I$. (ii) $I^\tau = 0$.
(iii) $(A^\sigma)^\sigma = A^\sigma$. (iv) $(A^\tau)^\tau = A^\tau$.
(v) $(A^\sigma B^\sigma)^\sigma = A^\sigma B^\sigma$. (vi) $(A^\tau B^\tau)^\tau = A^\tau B^\tau$.
(vii) $\|A^\sigma\| \leqq \|A\|$. (viii) $\|A^\tau\| \leqq 2\,\|A\|$.
(ix) $(\alpha A + \beta B)^\sigma = \alpha A^\sigma + \beta B^\sigma$.
(x) If $A_0 + A_1 + \cdots$ is a series of bounded operators of the form (1.1) whose partial sums form a Cauchy sequence in the operator norm, then $T = A_0 + A_1 + \cdots$ is a bounded linear operator of the form (1.1). Moreover, $A_0^\sigma + A_1^\sigma + \cdots$ and $A_0^\tau + A_1^\tau + \cdots$ converge in the operator norm, also $T^\sigma = A_0^\sigma + A_1^\sigma + \cdots$ and $T^\tau = A_0^\tau + A_1^\tau + \cdots$.

(xi) If $A = A_1 + A_2$ where $A_1 = A_1^\sigma$ and $A_2 = A_2^\tau$, then $A_1 = A^\sigma$ and $A_2 = A^\tau$.

We prove only (x). Since the space of bounded linear operators on $\mathcal{L}$ into $\mathcal{L}$ is a Banach space, T is a bounded linear operator on $\mathcal{L}$ and, therefore, must be of the form (1.1). Let $T_n = A_0 + A_1 + \cdots + A_n$. Since

$$\|T_n^\sigma - T^\sigma\| \leqq \|T_n - T\| \quad \text{and} \quad \|T^\tau - T_n^\tau\| \leqq 2\,\|T - T_n\|,$$

the second statement in (x) follows.

Note that properties (i)–(xi) say that any bounded linear operator A on $\mathcal{L}$ into $\mathcal{L}$ can be split uniquely into the sum of two operators A^σ and A^τ each of which is an element in a proper subspace of the Banach algebra of bounded linear operators on $\mathcal{L}$ into $\mathcal{L}$.

2. Definition of a semi-Markov process. In [7], Pyke and Schaufele presented a non-constructive definition of a semi-Markov process. It is this non-constructive definition that is used here, although we follow the form of the definition given in [10].

We take $\{X_t : t \geqq 0\}$ to be a separable process with a countable state space I. Let

$$\begin{aligned} Y_t &= t \quad \text{if} \quad X_s = X_t \quad \text{for all} \quad 0 \leqq s \leqq t, \\ &= t - \sup\,[s : 0 \leqq s \leqq t;\, X_s \neq X_t] \quad \text{otherwise}. \end{aligned}$$

If the two-dimensional process $\{(X_t, Y_t): t \geqq 0\}$ is a strong Markov process with stationary Borel measurable transition probabilities, then we say that $\{X_t : t \geqq 0\}$ is a semi-Markov process (SMP).

DEFINITION 2.1. Let

$$\begin{aligned} w_t &= \inf [s: s \geqq t; X_s \neq X_t], \quad \text{if} \quad X_u = X_t, \qquad \text{for} \quad 0 \leqq u \leqq t, \\ &= \inf [s: s \geqq t; X_s \neq X_t] - \sup [s: s \leqq t; X_s \neq X_t] \quad \text{otherwise;} \end{aligned}$$

$$F_i(t) = P[w_0 \leqq t \,|\, (X_0, Y_0) = (i, 0)].$$

For convenience we shall denote w_0 by w and $P[S \,|\, (X_t, Y_t) = (i, 0)]$ by $P_{i,0}[S]$ where S is a Borel subset of the state space of $\{(X_t, Y_t): t \geqq 0\}$.

For this paper we shall require that $F_i(t) \to 0$ as $t \to 0+$ for all $i \in I$. In this case, once the process enters a state it stays there for a positive length of time with probability one. That is, the process is a step process. We shall also assume that the process has right-continuous sample paths in order to guarantee that w is a stationary Markov time of the process.

DEFINITION 2.2. If $\{X_t : t \geqq 0\}$ is a right-continuous SMP for which $F_i(t) \to 0$ as $t \to 0+$ for all i in I, then we call $\{X_t : t \geqq 0\}$ a *semi-Markov step process* (SMSP).

DEFINITION 2.3. Let

$$\begin{aligned} Q_{ij}(t) &= P_{i,0}[w \leqq t \quad \text{and} \quad X_w = j] \quad \text{if} \quad i \neq j, \\ Q_{ii}(t) &= 0, \\ Z(t) &= \sup [X_s : 0 \leqq s \leqq t], \\ M_{ij}(t) &= P_{i,0}[Z(t) = j]. \end{aligned}$$

In this paper we take the point of view that the $Q_{ij}(t)$ are known and that the distributions of certain functionals on $\{X_t : t \geqq 0\}$ are to be solved in terms of them. This is an acceptable point of view even for a continuous parameter Markov chain since the $Q_{ij}(t)$ may be easily calculated from the transition probabilities $p_{ij}(t)$. In fact, in [3], p. 246 it is shown that

$$Q_{ij}(t) = p_{ij}(1 - e^{-c_i t}) \quad \text{for} \quad i \neq j,$$

where $c_i = \lim_{t \to 0+} (1 - p_{ii}(t))/t$ and $c_i p_{ij} = \lim_{t \to 0+} p_{ij}(t)/t$.

3. Semi-Markov processes. If for each t, $A(t)$ and $B(t)$ are matrices, then let

$$A(t) * B(t) = \left(\textstyle\sum_k \int_{[0,t]} B_{kj}(t - s)\, dA_{ik}(s)\right)$$

when this makes sense. In the context of this paper $A_{ij}(t)$ will be a non-decreasing function, and $B_{ij}(t)$ will be a Borel function. The above integrals are to be understood as Lebesgue-Stieltjes integrals with respect to the measures induced by the $A_{ik}(s)$'s.

The next theorem is one of the main results of this paper. There is given an equation involving the known functions $Q_{ij}(t)$ which is satisfied by the distribution $M_{ij}(t)$ of $Z(t)$. This is the generalization to semi-Markov processes of Baxter's results in [1] for discrete Markov processes.

THEOREM 3.1. *For a* SMSP, *let* $M(t) = (M_{ij}(t))$ *and* $Q(t) = (Q_{ij}(t))$. *Then for all* $t > 0$

$$M(t) = (\delta_{ij}(1 - F_i(t))) + (Q(t) * M(t))^\sigma \tag{3.1}$$

and

$$M_{ij}(t) \to \delta_{ij} \quad \text{as} \quad t \to 0+.$$

PROOF. We consider three cases:

Case 1, $j < i$. Clearly $M_{ij}(t) = 0$.

Case 2, $j > i$. If the process starts at i and has maximum $j > i$ over the interval $[0, t]$, then there must have been a transition in $[0, t]$. Since we are dealing with a step process we can suppose that the first jump is to $k = X_w$, where obviously $k \leqq j$.

Partitioning on the first jump, we have

$$M_{ij}(t) = P_{i,0}[Z_t = j] = \textstyle\sum_{k \leqq j} P_{i,0}[Z_t = j; X_w = k].$$

Let E_k denote expectations taken over $[X_w = k]$. Then

$$P_{i,0}[Z_t = j; X_w = k] = E_k(P_{i,0}[Z_t = j \,|(X_w, Y_w) = (k, 0)]).$$

By the strong Markov property and the stationarity of (X_t, Y_t), the process regenerates itself at jump times. Thus for $k \leqq j$,

$$P_{i,0}[Z_t = j \,|(X_w, Y_w) = (k, 0)] = P_{k,0}[Z_{t-w} = j] = M_{kj}(t - w),$$

and

$$P_{i,0}[Z_t = j; X_w = k] = E_k(M_{kj}(t - w)) = \textstyle\int_{[0,t]} M_{kj}(t - s)\, dQ_{ik}(s)$$

by a transformation theorem p. 342, [6].

Finally,

$$M_{ij}(t) = \textstyle\sum_{k \leqq j} \int_{[0,t]} M_{kj}(t - s)\, dQ_{ik}(s) = \sum_{k \epsilon I} \int_{[0,t]} M_{kj}(t - s)\, dQ_{ik}(s).$$

Case 3, $i = j$. If $Z_t = i$ and $X_0 = i$, there are two possibilities

(i) $w > t$. *Then* $P_{i,0}[w > t$ and $Z_t = i] = P_{i,0}[w > t] = 1 - F_i(t)$,

(ii) $w \leqq t$. Since $Z_t = i$, $X_w < i$, and $Z_{t-w} \leqq i$.

So by an argument similar to that given before

$$P_{i,0}[w \leqq t \quad \text{and} \quad Z_t = i] = \textstyle\sum_{j \leqq i} \sum_{k < i} \int_{[0,t]} M_{kj}(t - s)\, dQ_{ik}(s).$$

Since $Q_{ii}(t) = 0$, and $M_{kj}(t) = 0$ for $k > i$, we may write

$$P_{i,0}[w \leqq t \quad \text{and} \quad Z_t = i] = \textstyle\sum_{j \leqq i} \sum_{k \epsilon I} \int_{[0,t]} M_{kj}(t - s)\, dQ_{ik}(s).$$

It follows that

$$M_{ii}(t) = 1 - F_i(t) + \textstyle\sum_{j \leqq i} \sum_{k \epsilon I} \int_{[0,t]} M_{kj}(t - s) Q_{ik}(s).$$

By checking the definition of the σ operator we see that

$$(M_{ij}(t)) = (\delta_{ij}(1 - F_i(t))) + (Q(t) * M(t))^\sigma.$$

To show that $M_{ij}(t) \to \delta_{ij}$ as $t \to 0+$ we observe that

$$0 \leqq M_{ij}(t) \leqq P_{i,0}[w \leqq t] = F_i(t) \to 0 \quad \text{as} \quad t \to 0+ \qquad (i \neq j)$$

and

$$1 \geqq M_{ii}(t) \geqq P_{i,0}[w > t] = 1 - F_i(t) \to 1 \quad \text{as} \quad t \to 0+.$$

This completes the proof.

The convolutions appearing in Theorem 1 suggest that it might be convenient to work with Laplace transforms.

Define

$$\begin{aligned} m_{ij}(s) &= \int_{[0,\infty)} e^{-st} M_{ij}(t)\, dt \\ g_i(s) &= \int_{[0,\infty)} e^{-st}(1 - F_i(t))\, dt \qquad (3.2) \\ q_{ij}(s) &= \int_{[0,\infty)} e^{-st}\, dQ_{ij}(t) \end{aligned}$$

where the integrals above are to be understood as Lebesgue-Stieltjes integrals. Let $q(s) = (q_{ij}(s))$, $g(s) = (\delta_{ij} g_i(s))$, and $m(s) = (m_{ij}(s))$. We may now write Theorem 3.1 in the following convenient form

Corollary 3.1. *For a* SMSP

$$m(s) = g(s) + (q(s)m(s))^\sigma. \qquad (3.3)$$

Notice that we are now using a simple matrix product when we write $q(s)m(s)$.

We thank the referee for observing that $M_{ij}(t)$ is a right continuous function of bounded variation and that it is, therefore, uniquely determined by its Laplace transform. One may verify this in the following manner. For any integer n, let $A_n = \{n, n+1, \cdots,\} \cap I$ and let T_n be the hitting time of A_n. Then $P_{i,0}[Z(t) \leqq j] = P_{i,0}[T_{j+1} > t]$ is a right continuous non-increasing function of t. Since $M_{ij}(t) = P_{i,0}[T_{j+1} > t] - P_{i,0}[T_j > t]$, the verification is complete.

Although we have shown that $m(s)$ satisfies equation (3.3) in Corollary 3.1, we have no guarantee that $m(s)$ is the only family of matrices satisfying equation (3.3). The aim of the next theorem is to give conditions under which equation (3.3) uniquely determines $m(s)$. Note that $m(s)$ is a bounded operator on $\mathcal{L}$ into $\mathcal{L}$ for every $s > 0$.

Theorem 3.2. *If for some* $s_0 > 0$, $\|q(s_0)\| < 1$, *then* $m(s) = g(s) + (q(s)m(s))^\sigma$ *has a unique bounded solution for* $s \geqq s_0$.

Proof. Let $m(s)$ be a bounded solution of (3.3). Iterating equation (3.3) n times one obtains

$$m(s) = m_0(s) + m_1(s) + \cdots + m_n(s) + L_n(s),$$

where

$$m_0(s) = g(s), \qquad m_{n+1}(s) = (q(s)m_n(s))^\sigma,$$

and

$$L_n(s) = (q(s)(\cdots(q(s)m(s))\underbrace{^\sigma \cdots)^\sigma}_{n+1 \text{ times}}.$$

So, by properties (1.2)

$$\|m_n(s)\| \leqq \|q(s)\|^n \|g(s)\| \quad \text{and} \quad \|L_n(s)\| \leqq \|q(s)\|^{n+1} \|m(s)\|.$$

Since $\|g(s)\| < \infty$ and $\|q(s)\| \leqq \|q(s_0)\| < 1$ for $s \geqq s_0$, the series $\sum_{n=0}^{\infty} m_n(s)$ converges in the strong operator sense, and $\|L_n(s)\| \to 0$ as $n \to \infty$. Thus, $m(s) = \sum_{n=0}^{\infty} m_n(s)$ is the unique bounded solution of equation (3.3).

The condition $\|q(s_0)\| < 1$ for some s_0 implies that there are only a finite number of transitions in any interval $[0, t]$. To see this, let $p_{ij}(s) = \int_{[0,\infty]} P_{i,0}[X_t = j]e^{-st}\, dt$ and $p(s) = (p_{ij}(s))$, then the backward equation for a SMSP becomes

$$p(s) = g(s) + q(s)m(s). \tag{3.4}$$

If $\|q(s_0)\| < 1$, equation (3.4) has a unique bounded solution for $s \geqq s_0$. Thus, we know from [5] that the minimal solution is the unique substochastic solution of the backward equation, and with probability 1, there are only a finite number of transitions in any interval $[0, t]$.

If f is a function of t, let f' denote the derivative of f.

Corollary 3.2. *If for a* SMSP, $\sum_k Q'_{ik}(t) \leqq B$ *for all* i, *then for* $s > B$, $m(s)$ *in* (3.2) *is the unique bounded solution of equation* (3.3).

Proof. If $\sum_k Q'_{ik}(t) \leqq B$, then $\|q(s)\| \leqq B/s$. So for $s > B$, $\|q(s)\| < 1$ and Theorem 3.2 applies.

In the next theorem we prove an analog of Spitzer's identity for semi-Markov processes under assumptions which are very strong and are satisfied only in special cases. Yet, the method of proof suggests an approach to solving the general case in (3.3). This will be discussed after the next theorem is stated and proved.

Theorem 3.3. *For a* SMSP *suppose that* $F_i(t) = F(t)$ *for all* i *and that* $[q^k(s)]^\sigma q(s) = q(s)[q^k(s)]^\sigma$ *for all* $k \geqq 0$. *If* $\|q(s_0)\| < 1$ *for some* s_0, *then for* $s \geqq s_0$ *let*

$$L(s) = \log(I - q(s)) = -\textstyle\sum_1^\infty q^k(s)k^{-1}$$

and $\eta(s) = \int_0^\infty e^{-st}(1 - F(t))\, dt$.
Then for $s \geqq s_0$,

$$m(s) = \eta(s) \exp\left(\textstyle\sum_1^\infty (q^k(s))^\sigma k^{-1}\right).$$

Proof. The condition $[q^k(s)]^\sigma q(s) = q(s)[q^k(s)]^\sigma$ guarantees that $\exp(L(s)) = \exp(L(s)^\tau)\exp(L(s)^\sigma)$. By Theorem 3.2 we know that for $s \geqq s_0$, $m(s)$ is the unique bounded solution of $m(s) = g(s) + (q(s)m(s))^\sigma$ which in this case has the form

$$\eta(s)I = [(I - q(s))m(s)]^\sigma = [\exp(L(s))m(s)]^\sigma.$$

By verifying that $[\exp(L(s)^\tau)]^\sigma = I$, it is easily seen that for $s \geqq s_0$, $\eta(s)\exp(-L(s)^\sigma)$ is a bounded, and hence the unique bounded solution of the above equation. Thus,

$$m(s) = \eta(s) \exp\left(\textstyle\sum_{k=1}^{\infty} (q^k(s))^\sigma k^{-1}\right).$$

COROLLARY 3.3. *Let the* SMSP *have the integers for its state space and be spacially homogeneous* (i.e. $Q_{ik}(t) = Q(k - i, t)$). *If the* $Q_{ik}(t)$*'s are continuous and if* $\|q(s_0)\| < 1$ *for some* s_0, *then for* $s \geqq s_0$

$$m(s) = \eta(s) \exp\left(\textstyle\sum_1^{\infty} (q^k(s)/k)^\sigma\right).$$

PROOF. $F_i(t) = \sum_{k \varepsilon I} Q(k - i, t)$ which is independent of i.

We need only show that $q(s)(q^k(s))^\sigma = (q^k(s))^\sigma q(s)$, and we may apply Theorem 3.3 to get the result. We shall show this by proving that

$$(Q_k(t))^\sigma * Q(t) = Q(t) * (Q_k(t))^\sigma$$

where, $Q_1(t) = (Q_{ij}(t))$, and $Q_{n+1}(t) = Q(t) * Q_n(t)$.

Denote the elements of $(Q_k(t))^\sigma$ by $Q_k^\sigma(j - i, t)$. Then

$$Q(t) * (Q_k(t))^\sigma = \left(\textstyle\sum_{k \leqq j} \int_{[0,t]} Q_k^\sigma(j - k, t - s)\, dQ(k - i, s)\right).$$

By making the change of variable $z = j + i - k$, we obtain

$$\left(\textstyle\sum_{z \geqq i} \int_{[0,t]} Q_k^\sigma(z - i, t - s)\, dQ(j - z, s)\right).$$

Integration by parts gives

$$\left(\textstyle\sum_{z \geqq i} \int_{[0,t]} Q(j - z, t - s)\, dQ_k^\sigma(z - i, s)\right) = (Q_k(t))^\sigma * Q(t).$$

The construction in Theorem 3.3 is based on a Wiener-Hopf factorization. That is, we write (uniquely)

$$I - q(s) = \exp(L(s)^\tau) \exp(L(s)^\sigma) \tag{3.5}$$

where $(\exp(L(s)^\tau))^\sigma = I$ and $(\exp(L(s)^\sigma))^\sigma = \exp(L(s)^\sigma)$. That this factorization is the unique one of the type $I - q(s) = \exp(A(s)) \exp(B(s))$ where $A(s)^\tau = A(s)$ and $B(s)^\sigma = B(s)$ may be seen by taking logarithms in (3.5) and using property (xi) of (1.2).

The solution of Equation (3.3) given in Theorem 3.3 is then a multiple of the inverse of the right factor matrix on the right-hand side of (3.5). If $g(s)$ is not a multiple of the identity the situation becomes more complicated. In fact, we then want to find matrices $A(s)$ and $B(s)$ satisfying the conditions:

(1) $I - q(s) = (I + B(s))(I + A(s))$.

(2) $(I + A(s))$ has a bounded right inverse, $(I + A(s))^{-1}$.

(3) $(A(s))^\sigma = A(s)$.

(4) $B(s)$ is subdiagonal, and $\sum_{k=-\infty}^{i} B_{ik}(s) g_k(s) = 0$.

Then the (unique) bounded solution of (3.3) is $m(s) = (I + A(s))^{-1} g(s)$. Of course if $g(s)$ is a constant multiple of I, condition (4) becomes $(B(s))^\tau = B(s)$, and (1) is the familiar Wiener-Hopf factorization.

A simple example will serve to illustrate the above method and its difficulties.

Suppose that we have a two state semi-Markov process in which $Q_{ij}(t) = \int_0^t Q'_{ij}(s)\,ds$.

Then

$$q(s) = \begin{pmatrix} 0 & q_{01}(s) \\ q_{10}(s) & 0 \end{pmatrix},$$

and

$$g(s) = \begin{pmatrix} (1 - q_{01}(s))/s & 0 \\ 0 & (1 - q_{10}(s))/s \end{pmatrix}.$$

Of course, one can easily write down $m(s)$ from probabilistic considerations. Namely,

$$m(s) = \begin{pmatrix} (1 - q_{01}(s))/s & q_{01}(s)/s \\ 0 & 1/s \end{pmatrix}.$$

However, it is instructive to carry out the factorization described in (1)–(4). We have

$$I - q(s) = (I + B(s))(I + A(s))$$

where

$$B(s) = \begin{pmatrix} 0 & 0 \\ -q_{10}(s) & q_{10}(s)(1 - q_{01}(s))/(1 - q_{10}(s)) \end{pmatrix}$$

and

$$A(s) = \begin{pmatrix} 0 & -q_{01}(s) \\ 0 & -q_{10}(s) \end{pmatrix}.$$

Also,

$$(I + A(s))^{-1} = \begin{pmatrix} 1 & q_{01}(s)/(1 - q_{10}(s)) \\ 0 & 1/(1 - q_{10}(s)) \end{pmatrix}.$$

One can easily check that properties (1)–(4) hold and that

$$m(s) = (I + A(s))^{-1}g(s) = \begin{pmatrix} (1 - q_{01}(s))/s & q_{01}(s)/s \\ 0 & 1/s \end{pmatrix}.$$

4. Continuous parameter Markov chains. We now consider a continuous parameter Markov chain (MC) with stationary transition probabilities, $p_{ij}(t)$, as a special case of a SMP.

In this case,

$$F_i(t) = 1 - e^{-c_i t} \quad \text{and} \quad Q_{ij}(t) = \dot{p}_{ij}(1 - e^{-c_i t}),$$

where

$$c_i \geqq 0, \qquad p_{ij} \geqq 0, \qquad p_{ii} = 0 \quad \text{and} \quad \textstyle\sum_{j \in I} p_{ij} = 1.$$

The assumption that we are dealing with a step process means that there are no instantaneous states (i.e. $c_i < \infty$ for all i). Let us denote by $A = (a_{ij})$ the infinitesimal generator of the chain. Then,

$$\begin{aligned} a_{ij} &= p_{ij}c_i \qquad \text{if } i \neq j \\ &= -c_i \qquad \text{otherwise.} \end{aligned}$$

The analogs for MC's of the theorems of Section 3 are presented below. In spite of its simplicity, the following theorem appears not to be in the literature.

THEOREM 4.1. *For a stationary* MC *with a standard transition matrix and no instantaneous states,*

$$M'(t) = (AM(t))^\sigma \quad \text{where} \quad M'(t) = (M'_{ij}(t)). \tag{4.1}$$

PROOF. By Theorem 3.1,

$$(M_{ij}(t)) = (\delta_{ij}e^{-c_i t}) + (\textstyle\sum_k \int_{[0,t]} M_{kj}(t-s)p_{ik}c_i e^{-c_i s}\, ds)^\sigma.$$

Making the change of variable $y = t - s$, and taking derivatives termwise, we obtain, $(M'_{ij}(t)) = (AM(t))^\sigma$.

In terms of Laplace transforms, Theorem 4.1 becomes

COROLLARY 4.1. *For a stationary* MC *with a standard transition matrix and no instantaneous states,*

$$sm(s) = I + (Am(s))^\sigma. \tag{4.2}$$

THEOREM 4.2. *For a stationary* MC *with a standard transition matrix and no instaneous states,* $\|A\| = b < \infty$ *implies that for* $s > b$ *Equation* (4.2) *has a unique bounded solution* $m(s)$.

PROOF. For $s > b$, $\|A\|/s < 1$ and we may apply the proof of Theorem 3.2 to show that Equation 4.2 has a unique bounded solution.

Observe that the condition $\|A\| < b < \infty$ means that $c_i < b$ for all i. Now one may easily check that by Theorem 1 (II.19) of [3], almost all sample paths have only a finite number of transitions in any interval $[0, t]$.

THEOREM 4.3. *Consider a stationary* MC *with a standard transition matrix and no instantaneous states. If* $(A^k)^\sigma A = A(A^k)^\sigma$ *for all* $k \geqq 1$, *and if* $\|A\| = b < \infty$, *then for* $s > b$

$$m(s) = (1/s)\exp\left(\textstyle\sum_1^\infty (A^k)^\sigma/ks^k\right).$$

PROOF. Rewriting equation 4.2 as $I/s = ((I - A/s)m(s))^\sigma$, it is easily seen by a method similar to that used in Theorem 3.4 that $(1/s)\exp\left(\sum_1^\infty (A^k)^\sigma/ks^k\right)$ is the unique bounded solution of (4.2).

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