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ON THE DISTRIBUTION OF THE SUPREMUM FUNCTIONAL FOR SEMI-MARKOV PROCESSES WITH CONTINUOUS STATE SPACES

BY LAWRENCE D. STONE

Daniel H. Wagner, Associates

1. Introduction. Let $\{X_t:t \geqq 0\}$ be a separable stochastic process and $Z(t) = \sup [X_s:0 \leqq s \leqq t]$. In a previous paper [7] this author extended the results of Baxter [1] by characterizing the distribution of $Z(t)$ for a large class of Markov and semi-Markov processes with denumerable state spaces. These results were obtained by considering the time and place of the first jump.

The purpose of this paper is to extend the results of [7] to Markov and semi-Markov processes whose sample paths are step functions and whose state space may be an arbitrary subset of the real numbers. In order to do this we have defined semi-Markov jump processes in Section 2. These processes are a generalization of Markov jump processes as described in [4], p. 316, and semi-Markov processes as defined in [5] and [8]. Theorem 2.1 shows that these processes may also be analyzed by considering the time and place of the first jump.

In Section 3, the distribution of $Z(t)$ for a large class of semi-Markov jump processes is characterized by a recurrence relation involving operators. In the case where the process is homogeneous in space, an analog of Spitzer's identity (see [6]) is proved. In Section 4, an example is presented which shows how the recurrence relation can be used to guess the double transform of $Z(t)$ for processes which are homogeneous in space.

2. Semi-Markov jump processes. In this section we present the definition of a semi-Markov jump process and investigate briefly some of its properties.

Let $\{X_t:t \geqq 0\}$ be a separable stochastic process with random variables, X_t, defined on the probability space $(\Omega, \mathfrak{A}, P)$, and having their range in R, the real numbers. Define

$$\begin{aligned} Y_t &= t \quad \text{if} \quad X_s = X_t \qquad \text{for all} \quad 0 \leqq s \leqq t, \\ &= t - \sup [s:0 \leqq s \leqq t; X_s \neq X_t] \qquad \text{otherwise,} \end{aligned}$$

and

$$\alpha_t = \inf [s > t; X_s \neq X_t] \qquad \text{for} \quad t \geqq 0.$$

For convenience of notation let $\alpha = \alpha_0$.

Let $R^+ = [0, \infty)$; $\mathfrak{B}(R)$ and $\mathfrak{B}(R^+)$ denote the σ-algebras of Borel subsets of R and R^+ respectively. Define a two-dimensional process $\{(X_t, Y_t):t \geqq 0\}$; the

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state space of this process is a subset of $R \times R^+$. If Π is a member of $\mathcal{B}(R) \times \mathcal{B}(R^+)$, then for (x, y) in $R \times R^+$, let $P_{x,y}[(X_t, Y_t) \in \Pi]$ be a version of the conditional probability $P[(X_t, Y_t) \in \Pi \mid (X_0, Y_0) = (x, y)]$.

We say that $\{X_t : t \geqq 0\}$ is a *semi-Markov jump process* if the two-dimensonal process $\{(X_t, Y_t) : t \geqq 0\}$ has right-continuous sample paths, is a strong Markov process with stationary transition probabilities, and

$$P_{x,0}[0 < \alpha < \infty] = 1$$

for all x in R.

The intent of the condition on α is to guarantee that (with probability 1) the process remains a positive length of time in each state it enters (the finiteness of α is not essential, and with appropriate modifications α could be allowed to be infinite).

We wish to guarantee that α is a stopping time of the $\{(X_t, Y_t)\}$ process so that we may apply the strong Markov property to α. In order to assure that α is a stopping time, it is necessary to guarantee that for each t, $X_{\alpha_t} \neq X_t$, i.e., that jump points cannot accumulate from the right to a point at which no jump occurs. It is for this reason that we require the $\{Y_t\}$ process to be right continuous. The $\{X_t\}$ process then moves strictly by jumps with α_t the time to the first jump following time t and X_{α_t} the state to which the process jumps.

Since α is a stopping time and $\{X_t\}$ is right continuous, X_α is a random variable, and we may define a function q on $R \times R \times R^+$ by

$$q(x, y, t) = P_{x,0}\{X_\alpha \leqq y \quad \text{and} \quad \alpha \leqq t\}.$$

Note that

$$q(x, y, t) = \int_{(-\infty, y)} \int_{[0,t]} c(x, z, ds) k(x, dz)$$

where $c(x, z, s) = P_{x,0}[\alpha \leqq s \mid X_\alpha = z]$ and $k(x, z) = P_{x,0}[X_\alpha \leqq z]$.

Define

$$a(x, t) = \lim_{y \to \infty} q(x, y, t).$$

The condition that α be positive and finite may be expressed as $a(x, 0) = 0$ and $a(x, t) \to 1$ as $t \to \infty$ for all x in R.

The right-continuity of $\{(X_t, Y_t)\}$ implies that it is a measurable process; thus for a fixed set Π the transition functions are Borel measurable.

If the state space of a semi-Markov jump process is countable, then the process becomes a special case of a semi-Markov process as defined in [5] and [8]. In this case the functions $q(i, j, \cdot) - q(i, j-1, \cdot)$ become the functions $Q_{ij}(\cdot)$ of [5]. If $c(x, y, t) = 1 - e^{-\gamma(x)t} (0 < \gamma(x) < \infty)$, then $q(x, y, t) = (1 - e^{-\gamma(x)t}) k(x, y)$, and the semi-Markov jump process becomes a Markov jump process as described in [4], page 316.

In this paper we take the point of view that the functions a and q are known and that the distributions of functionals are to be solved in terms of them. For convenience, let $P_t(x, \Gamma) = P_{x,0}[X_t \in \Gamma]$ for x in R and Γ in $\mathcal{B}(R)$. Then

THEOREM 2.1. *For a semi-Markov jump process,*

$$P_t(x, \Gamma) = (1 - a(x, t))I_x(\Gamma) + \int_R \int_{[0,t]} P_{t-s}(z, \Gamma)c(x, z, ds)k(x, dz) \tag{2.1}$$

where $I_x(\Gamma) = 1$ *if* $x \varepsilon \Gamma$ *and* 0 *otherwise.*

PROOF. When $\alpha = s$, then $Y_\alpha = Y_s = 0$ and $X_\alpha = X_s$. Thus

$$\begin{aligned} P_{x,0}[X_t \varepsilon \Gamma; \alpha \leqq t] \\ &= \int_{R\to[0,t]} P_{x,0}[X_t \varepsilon \Gamma \mid X_\alpha = z; \alpha = s] P_{x,0}[X_\alpha \varepsilon dz, \alpha \varepsilon ds] \\ &= \int_R \int_{[0,t]} P_{z,0}[X_{t-s} \varepsilon \Gamma] c(x, z, ds)k(x, dz) \end{aligned} \tag{2.2}$$

the last line following from the strong Markov property applied to α. Also,

$$P_{x,0}[X_t \varepsilon \Gamma; \alpha > t] = (1 - a(x, t))I_x(\Gamma). \tag{2.3}$$

Combining (2.2) and (2.3) we have (2.1) and the theorem is proved.

We now discuss conditions under which Equation (2.1) uniquely determines the transition probabilities of a semi-Markov jump process.

For each x and Γ, define

$$P_t^{(0)}(x, \Gamma) = (1 - a(x, t))I_x(\Gamma)$$

and for $n \geqq 0$,

$$P_t^{(n+1)}(x, \Gamma) = P_t^{(0)}(x, \Gamma) + \int_R \int_{[0,t]} P_{t-s}^{(n)}(z, \Gamma)c(x, z, ds)k(x, dz).$$

Then in the manner of [4], one may show that $P_t^{(\infty)}(x, \Gamma) = \lim_{n\to\infty} P_t^{(n)}(x, \Gamma)$ exists and is the minimal non-negative solution of (2.1). Also, if $P_t^{(\infty)}(x, R) = 1$, then $P_t^{(\infty)}(x, \cdot)$ is the unique probability measure on $\mathcal{A}$ satisfying (2.1). From the definition of $P_t^{(\infty)}(x, \Gamma)$ one can see that it gives the probability of going from x to Γ in time t with only finitely many jumps. Thus $P_t^{(\infty)}(x, R) = 1$ corresponds to the situation where the process can make only a finite number of jumps in any finite interval $[0, t]$. Finally, observe that if there is a unique substochastic solution, then $P_t^{(\infty)}(x, R) = 1$.

3. The distribution of the supremum functional. In this section there is given an operator-theoretic characterization of the distribution of the supremum for a wide class of semi-Markov jump processes. The development of this section is modeled after that in [1]. Below we state, without proof, some properties of operators defined by measures.

Let L_∞ be the Banach space of bounded Borel measurable functions, f, defined on R with norm $\|f\| = \sup_x |f(x)|$. Let v be a real-valued function of two real variables. Let the variation of $v(x, \cdot)$ be denoted by Var $(v(x, \cdot))$,

Suppose v satisfies the following conditions:

(3.1)
- (i) for each y, $v(\cdot, y)$ is a Borel measurable function,
- (ii) for each x, Var $(v(x, \cdot)) < \infty$ and $v(x, \cdot)$ is right continuous,
- (iii) $\max_{-\infty<x<\infty}$ Var $(v(x, \cdot)) < \infty$,
- (iv) for each x, $\lim_{y\to-\infty} v(x, y) = 0$.

Then we may define an operator V from L_∞ to L_∞ by

$$(3.2) \qquad (Vf)(x) = \int_R f(y)v(x, dy)$$

with norm $\|V\| = \max_{-\infty<x<\infty} \mathrm{Var}\,(v(x, \cdot))$. We call v the kernel of the operator V.

If V_1 and V_2 are bounded operators of the form (3.2), with kernels v_1 and v_2, then V_1V_2 is an operator of the form (3.2) with kernel v given by

$$v(x, y) = \int_R v_2(z, y)v_1(x, dz).$$

We now define an operator, σ, on operators.

DEFINITION 3.1. If v is a function of two real variables then define v^σ by $v^\sigma(x, y) = 0$ if $y < x$ and $v^\sigma(x, y) = v(x, y)$ if $y \geqq x$.

DEFINITION 3.2. If V is an operator of the form (3.2) with kernel v, then define V^σ to be the operator with kernel v^σ.

Let $V^\tau = V - V^\sigma$, then both V^σ and V^τ are operators of the form (3.2). Let I be the identity operator, V_i, $i = 1, 2, \cdots$, be operators of the form (3.2), and α and β be real numbers. Then σ and τ have the following properties:

(3.3)

(i) $I^\sigma = I$. (ii) $I^\tau = 0$.

(iii) $(V_1^\sigma)^\sigma = V_1^\sigma$. (iv) $(V_1^\tau)^\tau = V_1^\tau$.

(v) $(V_1^\sigma V_2^\sigma)^\sigma = V_1^\sigma V_2^\sigma$. (vi) $(V_1^\tau V_2^\tau)^\tau = V_1^\tau V_2^\tau$.

(vii) $\|V_1^\sigma\| \leqq \|V_1\|$. (viii) $\|V_1^\tau\| \leqq 2\,\|V_1\|$.

(ix) $(\alpha V_1 + \beta V_2)^\sigma = \alpha V_1^\sigma + \beta V_2^\sigma$.

(x) If the partial sums of the series $V_1 + V_2 + V_3 + \cdots$ form a Cauchy sequence in the operator norm, then $T = V_1 + V_2 + V_3 \cdots$ is an operator of the form (3.2). Moreover, $V_1^\sigma + V_2^\sigma + \cdots$ and $V_1^\tau + V_2^\tau$ converge in the operator norm. Also, $T^\sigma = V_1^\sigma + V_2^\sigma + \cdots$ and $T^\tau = V_1^\tau + V_2^\tau + \cdots$.

(xi) If $V = V_1 + V_2$ where $V_1 = V_1^\sigma$ and $V_2 = V_2^\tau$, then $V_1 = V^\sigma$ and $V_2 = V^\tau$.

Properties (i)–(xi) say that any operator V of the form (3.2) can be split uniquely into the sum of two operators V^σ and V^τ each of which is an element in a proper subspace of the Banach algebra of bounded linear operators from L_∞ into L_∞.

Let $Z(t) = \sup\,[X(s):0 \leqq s \leqq t]$ and $f(t, x, y) = P_{x,0}[Z(t) \leqq y]$. Notice that for a fixed x and y, f is a decreasing function of t. Let $d(x, y)$ denote the kernel of the identity operator, I. The following theorem is one of the main results of this paper and is an analog of Theorem 3.1 in [7]. The theorem is in the form of a recurrence relation satisfied by f and involving the functions a and q defined in Section 2.

THEOREM 3.1. *Consider a semi-Markov jump process in which $f(\cdot, \cdot, y)$ is Borel measurable for each y in R. Then for all $t \geqq 0$,*

$$(3.4) \quad f(x, y, t) = d(x, y)(1 - a(x, t)) + [\int_R \int_{[0,t]} f(z, y, t - s)c(x, z, ds)k(x, dz)]^\sigma.$$

PROOF. For $y < x$, $f(x, y, t) = 0$. Suppose $y \geqq x$. Then

$P_{x,0}[Z(t) \leqq y] = P_{x,0}[Z(t) \leqq y \text{ and } \alpha > t] + P_{x,0}[Z(t) \leqq y \text{ and } \alpha \leqq t]$.

Now,

$$P_{x,0}[Z(t) \leqq y \quad \text{and} \quad \alpha > t] = 1 - a(x, t),$$

and recalling that $\alpha = s$ implies that $Y_s = 0$ and $X_\alpha = X_s$, we have

$$\begin{aligned} P_{x,0}[Z(t) \leqq y \quad &\text{and} \quad \alpha \leqq t] \\ &= \int_{R\to[0,t]} P_{x,0}[Z(t) \leqq y \mid X_\alpha = z; \alpha = s] P_{x,0}[X_\alpha \,\varepsilon\, dz, \alpha \,\varepsilon\, ds] \\ &= \int_R \int_{[0,t]} f(z, y, t - s) c(x, z, ds) k(x, dz) \end{aligned}$$

where the last line follows from the strong Markov property applied to α. Thus for $y \geqq x$,

$$f(x, y, t) = 1 - a(x, t) + \int_R \int_{[0,t]} f(z, y, t - s) c(x, z, ds) k(x, dz).$$

By checking the definition of the operator σ, we see that for each $t \geqq 0$,

$$f(x, y, t) = d(x, y)(1 - a(x, t)) + \left[\int_R \int_{[0,t]} f(z, y, t - s) c(x, z, ds) k(x, dz)\right]^\sigma,$$

and the theorem is proved.

It is convenient to recast Theorem 3.1 in terms of Laplace transforms. Since $f(x, y, \cdot)$ is right-continuous and monotone decreasing, it is uniquely determined by its Laplace transform.

For each $t \geqq 0$, let $V(t)$ be an operator of the form (3.2) with kernel $v(\cdot, \cdot, t)$ satisfying (3.1) and such that $\|V(t)\| \leqq C$ for all $t \geqq 0$. If for each x and y, $v(x, y, \cdot)$ is a bounded Borel measurable function, then let

$$\hat{v}(x, y, \lambda) = \int_{[0,\infty)} e^{-\lambda t} v(x, y, t)\, dt.$$

If $\hat{V}(\lambda)$ is the operator of the form (3.2) with kernel function $\hat{v}(\cdot, \cdot, \lambda)$, then one can easily show that for $\lambda > 0$, $\|\hat{V}(\lambda)\| \leqq C/\lambda$. To do this it is sufficient to consider operators with kernels such that $v(x, \cdot, t)$ is a non-decreasing function for each x and $t \geqq 0$. Under these conditions, we have for each x and $\lambda > 0$,

$$\begin{aligned} \operatorname{Var}(\hat{v}(x, \cdot, \lambda)) &= \lim_{y\to\infty} \hat{v}(x, y, \lambda) = \lim_{y\to\infty} \int_{[0,\infty)} e^{-\lambda t} v(x, y, t)\, dt \\ &= \int_{[0,\infty)} e^{-\lambda t} (\lim_{y\to\infty} v(x, y, t))\, dt \leqq \int_{[0,\infty)} e^{-\lambda t} C\, dt = C/\lambda. \end{aligned}$$

It follows that $\|V(\lambda)\| \leqq C/\lambda$.

Let

$$\hat{f}(x, y, \lambda) = \int_{[0,\infty)} e^{-\lambda t} f(x, y, t)\, dt,$$

$$\hat{q}(x, y, \lambda) = \int_{[0,\infty)} e^{-\lambda t} q(x, y, dt),$$

$$\hat{a}(x, \lambda) = \int_{[0,\infty)} e^{-\lambda t} a(x, dt).$$

By the above discussion $f(\cdot, \cdot, \lambda)$ is a kernel function, and it is easy to verify that

$q(\cdot, \cdot, \lambda)$ is one also. For each $t \geqq 0$, let $F(t)$, $\hat{F}(\lambda)$, $Q(t)$, and $\hat{Q}(\lambda)$ be the operators with kernel functions $f(\cdot, \cdot, t)$, $\hat{f}(\cdot, \cdot, \lambda)$, $q(\cdot, \cdot, t)$, and $\hat{q}(\cdot, \cdot, \lambda)$ respectively. Denote by $\hat{A}(\lambda)$ and $\hat{B}(\lambda)$ the operators with kernel functions defined by $d(x, y)\hat{a}(x, \lambda)$ and $d(x, y)(1 - \hat{a}(x, \lambda))/\lambda$ respectively. Finally, let K be the operator with kernel k.

Let us note for use in the proof of the corollaries below that

$$\hat{q}(x, y, \lambda) = \int_{(-\infty, y]} \hat{c}(x, z, \lambda)k(x, dz)$$

where

$$\hat{c}(x, z, \lambda) = \int_{[0, -\infty)} e^{-\lambda t} c(x, z, dt).$$

Corollary 3.1. *Under the hypotheses of Theorem* 3.1

$$\hat{F}(\lambda) = \hat{B}(\lambda) + [\hat{Q}(\lambda)\hat{F}(\lambda)]^{\sigma}. \tag{3.5}$$

If, in addition, α is stochastically independent of X_α, then

$$\hat{F}(\lambda) = \hat{B}(\lambda) + [\hat{A}(\lambda)K\hat{F}(\lambda)]^{\sigma}. \tag{3.6}$$

Proof. For $y \geqq x$

$$f(x, y, t) = 1 - a(x, t) + \int_R \int_{[0,t]} f(z, y, t - s)c(x, z, ds)k(x, dz),$$

and

$$\hat{f}(x, y, \lambda) = [(1 - \hat{a}(x, \lambda))/\lambda] + \int_R \hat{f}(z, y, \lambda)\hat{c}(x, z, \lambda)\, d(x, dz).$$

Since $f(x, y, t) = 0$ for $y < x$, we may write for all x and y

$$\hat{f}(x, y, \lambda) = d(x, y)[(1 - \hat{a}(x, \lambda))/\lambda] + [\int_R \hat{f}(z, y, \lambda)\hat{c}(x, z, \lambda)k(x, dz)]^{\sigma}.$$

Thus

$$\hat{F}(\lambda) = \hat{B}(\lambda) + [\hat{Q}(\lambda)\hat{F}(\lambda)]^{\sigma}.$$

Suppose that α is independent of X_α, then for each x and Γ, $c(x, \cdot, \Gamma)$ is constant, and one may easily verify that

$$q(x, y, t) = \int_{(-\infty, y]} \int_{[0,t]} a(x, ds)k(x, dz).$$

It then follows that $\hat{Q}(\lambda) = \hat{A}(\lambda)K$ and that

$$\hat{F}(\lambda) = \hat{B}(\lambda) + [\hat{A}(\lambda)K\hat{F}(\lambda)]^{\sigma}.$$

This finishes the proof of the corollary.

Corollary 3.1 gives a recurrence relation which is satisfied by $\hat{F}(\lambda)$, and the next theorem gives a condition under which $\hat{F}(\lambda)$ is the unique bounded solution of (3.5).

Theorem 3.2. *Suppose that the hypotheses of Theorem* 3.1 *are satisfied. If for some λ_0, $\|Q(\lambda_0)\| < 1$, then for all $\lambda > \lambda_0$, Equation* (3.5) *has a unique bounded solution.*

Proof. Let $\hat{F}(\lambda)$ be a bounded solution of (3.5). Iterating (3.5) n times one obtains

$$\hat{F}(\lambda) = \hat{F}_0(\lambda) + \hat{F}_1(\lambda) + \cdots + \hat{F}_n(\lambda) + \hat{L}_n(\lambda)$$

where

$$\hat{F}_0(\lambda) = \hat{B}(\lambda),$$
$$\hat{F}_{n+1}(\lambda) = (\hat{Q}(\lambda)\hat{F}_n(\lambda))^\sigma \qquad \text{for} \quad n \geqq 0,$$

and

$$\hat{L}_n(\lambda) = (\hat{Q}(\lambda)(\cdots(\hat{Q}(\lambda)\hat{F}(\lambda))^\sigma \underbrace{\cdots)^\sigma}_{n+1 \text{ times.}}$$

By properties (3.3),

$$\|\hat{F}_n(\lambda)\| \leqq \|\hat{Q}(\lambda)\|^n \|\hat{B}(\lambda)\|,$$

and

$$\|\hat{L}_n(\lambda)\| \leqq \|\hat{Q}(\lambda)\|^{n+1} \|\hat{F}(\lambda)\|.$$

Since $\|\hat{B}(\lambda)\| < \infty$, $\|\hat{F}(\lambda)\| < \infty$, and $\|\hat{Q}(\lambda)\| < 1$ for $\lambda > \lambda_0$, we have that the series $\sum_{n=0}^{\infty} \hat{F}(\lambda)$ converges in the strong operator sense and $\|\hat{L}_n(\lambda)\| \to 0$ as $n \to \infty$. Thus

$$\hat{F}(\lambda) = \sum_{n=0}^{\infty} \hat{F}_n(\lambda)$$

is the unique bounded solution of Equation (3.5).

COROLLARY 3.2. *If the hypotheses of Theorem* 3.1 *hold and if* α *is stochastically independent of* X_α, *then* $\|\hat{A}(\lambda)\| < 1$ *for some* λ_0 *implies* (3.6) *has a unique bounded solution for* $\lambda > \lambda_0$.

PROOF. Since $\hat{Q}(\lambda) = \hat{A}(\lambda)K$ and $\|K\| \leqq 1$, we have that $\|\hat{Q}(\lambda)\| \leqq \|\hat{A}(\lambda)\|$, and the corollary follows from Theorem 3.2.

Let us return to the discussion in Section 2 and define $p(x, y, t) = P_t(x, (-\infty, y))$. Note that $p(x, y, \cdot)$ is right-continuous and therefore uniquely determined by its Laplace transform. Let

$$p(x, y, \lambda) = \int_{[0,\infty)} e^{-\lambda t} p(x, y, t)\, dt$$

and $\hat{P}(\lambda)$ be the operator with kernel $\hat{p}(\cdot, \cdot, \lambda)$. Equation (2.1) becomes

$$\hat{P}(\lambda) = \hat{B}(\lambda) + \hat{Q}(\lambda)\hat{P}(\lambda). \tag{3.7}$$

The conditions of Theorem 3.2 or Corollary 3.2 guarantee that (3.7) has a unique bounded solution and thus that (2.1) has a unique substochastic solution. From the discussion in Section 2, we can now say that the conditions of Theorem 3.2 or Corollary 3.2 imply that the semi-Markov jump process makes only a finite number of jumps in any finite interval of time.

In the case that the process is homogeneous in space (i.e., $q(x, y, t) = q(y - x, t)$, $a(x, t) = a(t)$ and $k(x, y) = k(y - x)$), we may avoid operators and use Fourier transforms. Let

$$\mu(\theta, \lambda) = \int_R e^{i\theta y} \hat{f}(dy, \lambda),$$
$$\xi(\theta, \lambda) = \int_R e^{i\theta y} \hat{q}(dy, \lambda),$$

and

$$\varphi(\theta) = \int_R e^{i\theta y} k(dy).$$

Note that if

$$\psi(\theta) = \int_R e^{i\theta y}\, dG(y)$$

for some right continuous distribution function G, then

$$\psi^\sigma(\theta) = \int_{[0,\infty)} e^{i\theta y}\, dG(y) + G(0).$$

THEOREM 3.3. *If a semi-Markov jump process is homogeneous in space and satisfies the conditions of Theorem* 3.1, *then for* $\lambda > 0$, $\mu(\theta, \lambda)$ *is the unique bounded solution of*

(3.8) $$\mu(\theta, \lambda) = [(1 - a(\lambda))/\lambda] + (\mu(\theta, \lambda)\xi(\theta, \lambda))^\sigma.$$

In fact,

(3.9) $$\mu(\theta, \lambda) = [(1 - \hat{a}(\lambda))/\lambda] \exp[-\log(1 - \xi(\theta, \lambda))^\sigma].$$

PROOF. The proof that $\mu(\theta, \lambda)$ is the unique bounded solution of (3.8) may be accomplished in a manner similar to that in Theorems 3.1 and 3.2. To show (3.9) we use a Wiener-Hopf type factorization. For $\lambda > 0$, let

$$L(\theta, \lambda) = \log(1 - \xi(\theta, \lambda)).$$

Rewrite equation (3.8) as

$$[\mu(\theta, \lambda) \exp(L(\theta, \lambda))]^\sigma = [\mu(\theta, \lambda)[1 - \xi(\theta, \lambda)]]^\sigma = [(1 - \hat{a}(\lambda))/\lambda].$$

One may easily verify that $\exp(L(\theta, \lambda)) = \exp(L(\theta, \lambda)^\sigma) \exp(L(\theta, \lambda)^\tau)$, that $[\exp(L(\theta, \lambda)^\sigma)]^\sigma = \exp(L(\theta, \lambda)^\sigma)$, and that $[\exp(L(\theta, \lambda)^\tau)]^\sigma = 1$. Then one can see that $\mu(\theta, \lambda) = \exp(-L(\theta, \lambda)^\sigma)$ is a bounded (and hence the unique bounded) solution of (3.8). This finishes the proof.

Writing equation (3.7) as

$$\mu(\theta, \lambda) = [(1 - \hat{a}(\lambda))/\lambda] \exp\left(\textstyle\sum_{k=1}^{\infty} [\xi^k(\theta, \lambda)]^\sigma k^{-1}\right),$$

one can see that (3.9) is an analog of Spitzer's identity (see [6]).

COROLLARY 3.3. *If the conditions of Theorem* 3.3 *are satisfied and if* α *is stochastically independent of* X_α, *then* $\mu(\theta, \lambda)$ *is the unique bounded solution of*

(3.10) $$\mu(\theta, \lambda) = [(1 - \hat{a}(\lambda))/\lambda] + (\mu(\theta, \lambda)\varphi(\theta)\hat{a}(\lambda))^\sigma,$$

and may be written as

(3.11) $$\mu(\theta, \lambda) = [(1 - \hat{a}(\lambda))/\lambda] \exp(-\log(1 - \varphi(\theta)\hat{a}(\lambda))^\sigma).$$

Proof. We need only note that if the distribution of α is stochastically independent of X_α, then $\xi(\theta, \lambda) = \varphi(\theta)\hat{a}(\lambda)$ and the corollary follows directly from Theorem 3.3.

4. Examples. In this section we present a generalization of Example 2 in [2].

Let $\{Y_n\}, n \geqq 1$, be a sequence of independent random variables each having distribution $\Pr\{Y_n = k\} = p_k$, $k = 0, \pm 1, \pm 2, \cdots$ such that $p_k = 0$ for $k = 2, 3, 4, \cdots$ and $p_1 > 0$. Let $N(t)$ be the number of renewals having occurred over $(0, t]$ in a renewal process with renewal distribution function $a(t)$. It is assumed that a renewal has occurred at time 0. Let $Y_0 = 0$, then $X(t) = \sum_{n=0}^{N(t)} Y_n$ is a Markov jump process which is homogeneous in space and for which $\xi(\theta, \lambda) = \hat{a}(\lambda)\varphi(\theta)$. In the terminology of Corollary 3.3,

$$\hat{a}(\lambda) = \int_{[0,\infty)} e^{-\lambda t} a(dt);$$

and

$$\varphi(\theta) = \sum_{k=-1}^{\infty} p_{-k} e^{-ik\theta}.$$

For $\lambda > 0$, let $\rho(\lambda)$ be the unique positive solution of

$$(1/\hat{a}(\lambda)) = \sum_{k=-1}^{\infty} p_{-k} e^{-k\rho(\lambda)}.$$

Then one may check that

$$\mu(\lambda, \theta) = \lambda^{-1}(1 - e^{-\rho(\lambda)}) \sum_{k=0}^{\infty} e^{-k\rho(\lambda)+ik\theta}$$

is a bounded solution of (3.10) and thus that $\mu(\theta, \lambda)$ is the double transform of $f(y, t) = \Pr\{\sup_{0 \leqq s \leqq t} X(s) \leqq y\}$.

We present two special cases in which one may invert $\mu(\theta, \lambda)$ to find $f(y, t)$ explicitly. The first case is that of coin tossing at random times. This is well known and has been obtained previously by other methods (e.g., see [2]). In our terminology we have $\Pr\{Y_n = 1\} = \Pr\{Y_n = -1\} = \frac{1}{2}$ and $a(t) = 1 - e^{-\beta t}$. Also,

$$\rho(\lambda) = \cosh^{-1}(1/\hat{a}(\lambda)) = \cosh^{-1}((\lambda + \beta)/\beta).$$

So the Laplace transform of $f(n, \cdot)$ is

$$\lambda^{-1}(1 - e^{-\rho(\lambda)}) \sum_{k=0}^{n} e^{-k\rho(\lambda)} = \lambda^{-1} - \lambda^{-1} e^{-(n+1)\rho(\lambda)}$$
$$= \lambda^{-1} - \lambda^{-1}[((\lambda + \beta)/\beta) - (((\lambda + \beta)/\beta)^2 - 1)^{\frac{1}{2}}]^{n+1}.$$

Thus we obtain

$$\Pr\{\sup_{0 \leqq s \leqq t} X(s) \leqq n\} = 1 - (n + 1) \int_0^t e^{-\beta s} s^{-1} I_{n+1}(\beta s)\, ds$$

where I_n is the nth order modified Bessel function of the first kind.

The second case is the same as the first except that $a(t)$ is a gamma distribution function with density $\beta^{\frac{1}{2}} x^{-\frac{1}{2}} e^{-\beta x}/\pi^{\frac{1}{2}}$. Then $\rho(\lambda) = \cosh^{-1}(((\lambda + \beta)/\beta)^{\frac{1}{2}})$, and the Laplace transform of $f(n, \cdot)$ is

$$\lambda^{-1} - \lambda^{-1}[((\lambda + \beta)/\beta)^{\frac{1}{2}} - (\lambda/\beta)^{\frac{1}{2}}]^{n+1}$$
$$= \lambda^{-1} - \lambda^{-1}[(2\lambda/\beta) + 1 - (((2\lambda/\beta) + 1) - 1)^{\frac{1}{2}}]^{\frac{1}{2}(n+1)}.$$

Thus

$$\Pr\{\sup_{0 \leqq s \leqq t} X(s) \leqq n\} = 1 - \tfrac{1}{2}(n+1)\int_0^t e^{-\beta s/2} s^{-1} I_{\frac{1}{2}(n+1)}(\beta s/2)\, ds.$$

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