Bayes Derivation of Multitarget Intensity Filters

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Abstract* – *The multitarget intensity filter is derived from a Bayesian first principles approach. The multitarget and measurement models are assumed to be Poisson point processes. The Bayes multitarget posterior probability density function is first defined on the Poisson event space, and then reformulated in terms of the intensity functions that characterize all Poisson point processes. It is shown that the predicted multitarget and predicted measurement processes are Poisson. However, the multitarget Bayes posterior probability density is not that of a Poisson point process. It is shown that all the single-target marginal probability density functions of the multitarget posterior probability density are identical. Consequently, the multitarget Bayes posterior probability density is approximated as the product of its marginal probability densities. Maximum likelihood determines the scale factor that converts marginal probability density to posterior multitarget intensity. This posterior multitarget intensity defines the information updated multitarget Poisson point process.

Keywords: Multitarget tracking, intensity filter, Poisson point process, data association, PHD filter.[1]

1 Introduction

A self-contained derivation of a multitarget intensity filter from Bayes principles is presented. The multitarget intensity is the intensity of a Poisson point process. The Bayes information updated multitarget probability density is derived and shown *not* to correspond to that of a Poisson point process. However, its single-target marginal probability density functions (pdfs) are all identical. This leads to a Poisson point process approximation, and this approximation completes the Bayes recursion.

One on the main purposes of this paper to show that multitarget intensity filters can be understood in essentially elementary terms. The derivation assumes familiarity with single target Bayesian filtering and with Poisson point processes at an elementary level.

The intensity filter obtained here is very similar to the PHD (Probability Hypothesis Density) filter [1]. However, the target birth and measurement clutter processes that are assumed *a priori* in [1] are estimated here. The PHD filter is obtained by replacing these estimates with known birth and clutter process intensities.

The first papers to provide an alternative derivation of intensity filters to that provided in [1] are the series of papers by Erdinc, Willett, and Bar-Shalom [2,3,4]. Their physical-space approach is intuitively appealing in that it models the flow between bins in a discretized model of target state space and then recovers the PHD filter in the limit as bin size goes to zero. Their approach is significantly different from that taken by the present paper, which does not use discretization and is based directly on the mechanics of a Bayes formulation. The papers [2,3,4] also include the CPHD (Cardinalized PHD) filter, which is not discussed in the present paper, and they compare and contrast the PHD filter with other approaches to multitarget tracking.

2 Bayes method for Poisson models

The multitarget state comprises the number and states of the targets in target state space, $\mathcal{S}$. The multitarget state process is characterized as a Poisson process on $\mathcal{S}$. The state space $\mathcal{S}$ is taken to be a specified bounded subset of $\mathbb{R}^n$, $n \geq 1$; however, the methods presented here can be generalized to any space on which a Poisson process can be defined. The multitarget Poisson process is assumed to be the linear superposition of two Poisson processes – a target motion process and a target birth process. Target death is incorporated into the target motion process, but target death is not itself a Poisson process. Let Ξ_k and B_k denote the random variables of the multitarget state and birth processes at time t_k. The intensity of B_k, denoted by $b_k(x_k)$, is assumed known *a priori*.

Multitarget measurements are given as an ordered list of data points in the measurement space, $\mathcal{Z}$. The order of the points in the list is uninformative, so the list is a set. The data set is characterized as a realization of a Poisson point process on $\mathcal{Z}$. The multitarget data process is assumed to be the linear superposition of two Poisson processes – a target measurement process and a clutter measurement process. Let Υ_k and Λ_k denote the variables of the target data and clutter processes at time

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t_k. The intensity of Λ_k, denoted by $\lambda_k^{clutter}(z_k)$, is assumed known *a priori*.

The conditioning assumptions for the Poisson variables are implicit in the Bayes net depicted in Figure 1 of Appendix 1. The conditioning differs from the usual single target conditioning because the target birth and measurement clutter terms act as *a priori* variables.

A rather cumbersome conditioning notation is used throughout this paper to ensure there is no confusion whatever about the meaning and dependencies of the random variables. While tiresome in places, the added clarity is well worthwhile.

3 Predicted target intensity

The total target Poisson point process at time t_{k-1} is

$$\Xi_{k-1} \mid \mathrm{B}_1, \dots, \mathrm{B}_{k-1}, \Upsilon_1, \dots, \Upsilon_{k-1}, \Lambda_1, \dots, \Lambda_{k-1}. \qquad (1)$$

This variate includes both detected and undetected targets at time t_{k-1}. Its intensity is denoted by $f_{k-1|k-1}(x_{k-1})$ and is, by induction, assumed given. By definition of the intensity function, $\int_S f_{k-1|k-1}(x_{k-1})\,ds$ equals the expected number of targets in S for $S \in \mathcal{S}$. The pdf of (1) evaluated at the realization

$$\xi_{k-1} = \left(m^{\xi_{k-1}}, x_1^{\xi_{k-1}}, \dots, x_{m^{\xi_{k-1}}}^{\xi_{k-1}}\right) \in \mathscr{E}(\mathcal{S}) \qquad (2)$$

of the multitarget state at time t_{k-1} is given by

$$\frac{e^{-\int_S f_{k-1|k-1}(x)dx}}{m^{\xi_{k-1}}!} \prod_{j=1}^{m^{\xi_{k-1}}} f_{k-1|k-1}\left(x_j^{\xi_{k-1}}\right). \qquad (3)$$

The predicted total target process is the sum of two processes, a current-target process and a new-target process. The current-target process is computed first.

3.1 Predicted current-target intensity

The predicted current-target variate at time t_k, denoted

$$\Xi_k \mid \mathrm{B}_1, \dots, \mathrm{B}_{k-1}, \Upsilon_1, \dots, \Upsilon_{k-1}, \Lambda_1, \dots, \Lambda_{k-1}, \qquad (4)$$

is defined by subjecting every realization of the total target process (1) to two successive transformations. The first is survival filtering, and the second is target motion.

Let $d_{k-1}(x)$ be the probability that a target at x at time t_{k-1} does not transition to time t_k, i.e., dies. The Bernoulli variables $\{d_{k-1}(x) : x \in \mathcal{S}\}$ are assumed independent. Every realization of the Poisson variate (4) with intensity $f_{k-1|k-1}(x_{k-1})$ is subjected to survival filtering via the independent Bernoulli variables $\{1 - d_{k-1}(x) : x \in \mathcal{S}\}$. For example, the point $x_j^{\xi_{k-1}}$ in the realization (2) is retained with probability $1 - d_{k-1}\left(x_j^{\xi_{k-1}}\right)$ and deleted with probability $d_{k-1}\left(x_j^{\xi_{k-1}}\right)$. The points remaining after survival filtering form a Poisson process with intensity (see Appendix 2)

$$f_{k-1|k-1}^{Surviving}(x_{k-1}) = \left(1 - d_{k-1}(x_{k-1})\right) f_{k-1|k-1}(x_{k-1}). \qquad (5)$$

Let $\Psi_{X_k|X_{k-1}}(x \mid y)$ denote the probability that a target at y at time t_{k-1} transitions to x at time t_k. The multitarget transition probability function is defined by

$$\Phi_{\Xi_k|\Xi_{k-1}}(\xi_k \mid \xi_{k-1}) = \begin{cases} \prod_{j=1}^{m} \Psi_{X_k|X_{k-1}}\left(x_j^{\xi_k} \mid x_j^{\xi_{k-1}}\right), & \text{if } m^{\xi_k} = m^{\xi_{k-1}} \equiv m, \\ 0, & \text{if } m^{\xi_k} \neq m^{\xi_{k-1}}, \end{cases} \qquad (6)$$

where $\xi_k = \left(m^{\xi_k}, x_1^{\xi_k}, \dots, x_{m^{\xi_k}}^{\xi_k}\right) \in \mathscr{E}(\mathcal{S})$. Every realization of the surviving target process with intensity $f_{k-1|k-1}^{Surviving}(x_{k-1})$ is independently subjected to the transition function (6). The points transitioned in this way form the predicted current-target process. From Appendix 3, this process is Poisson and its intensity function is given by

$$f_{k|k-1}^{Current}(x_k) = \int_{\mathcal{S}} \Psi_{X_k|X_{k-1}}(x_k \mid x_{k-1})\, f_{k-1|k-1}^{Surviving}(x_{k-1})\, dx_{k-1}. \qquad (7)$$

Substituting (5) into (7) gives the predicted current-target intensity at time t_k as

$$f_{k|k-1}^{Current}(x_k) = \int_{\mathcal{S}} \Psi_{X_k|X_{k-1}}(x_k \mid x_{k-1}) \left(1 - d_{k-1}(x_{k-1})\right) f_{k-1|k-1}(x_{k-1})\, dx_{k-1}. \qquad (8)$$

3.2 Predicted total target intensity

The predicted total target variate that includes birth contributions at time t_k is

$$\Xi_k \mid \mathrm{B}_1, \dots, \mathrm{B}_k, \Upsilon_1, \dots, \Upsilon_{k-1}, \Lambda_1, \dots, \Lambda_{k-1}. \qquad (9)$$

The variate (9) is defined to be the superposition of the birth process B_k and the current-target process (4). The predicted total target variate (9) is Poisson because the superposition of Poisson variates is Poisson, and its intensity is the sum of the intensities of the superposed variates. Thus,

$$\begin{aligned} f_{k|k-1}(x_k) &= b_k(x_k) + f_{k|k-1}^{Current}(x_k) \\ &= b_k(x_k) + \int_{\mathcal{S}} \Psi_{X_k|X_{k-1}}(x_k \mid x_{k-1}) \left(1 - d_{k-1}(x_{k-1})\right) f_{k-1|k-1}(x_{k-1})\, dx_{k-1} \end{aligned} \qquad (10)$$

is the predicted total target intensity at time t_k.

4 Predicted measurement intensity

The predicted measurement process is the sum of two processes, namely, a target-originated measurement process and a clutter-originated measurement process. The target-originated process is computed first.

4.1 Target-originated data intensity

The predicted target-originated measurement variate at time t_k is

$$\Upsilon_k \mid \mathrm{B}_1,...,\mathrm{B}_k,\Upsilon_1,...,\Upsilon_{k-1},\Lambda_1,...,\Lambda_{k-1}\,. \qquad (11)$$

The variate (11) characterizes data generated by targets, but not clutter. Let $P_k^D(x)$ be the probability that a target at x at time t_k is detected. Then the predicted detected-target process at time t_k is Poisson and its intensity is

$$f_{k|k-1}^{Detected}(x_k) = P_k^D(x_k)\, f_{k|k-1}(x_k)\,. \qquad (12)$$

Let $p_{Z_k|X_k}(z \mid x)$ be the probability of a measurement $z \in \mathcal{Z}$ conditioned on a detected target at x at time t_k. Let

$$\upsilon_k = (m, z_1, ..., z_m) \in \mathscr{E}(\mathcal{Z}) \qquad (13)$$

be a measurement generated by the detected-target process at time t_k. The conditional probability of the measurement (13) is defined by

$$p_{\Upsilon_k|\Xi_k}(\upsilon_k \mid \xi_k) = \begin{cases} \prod_{j=1}^{m} p_{Z_k|X_k}\left(z_j \mid x_j^{\xi_k}\right), & \text{if } m^{\xi_k} = m \\ 0, & \text{if } m^{\xi_k} \neq m. \end{cases} \qquad (14)$$

By Bayes Theorem the pdf of the variate (11) is (suppressing cumbersome conditioning variables in the argument)

$$p_{\Upsilon_k|\mathrm{B}_1,...,\mathrm{B}_k,\Upsilon_1,...,\Upsilon_{k-1},\Lambda_1,...,\Lambda_{k-1}}(\upsilon_k) = \sum_{\xi_k \in \mathscr{E}(\mathcal{S})} p_{\Upsilon_k|\Xi_k}(\upsilon_k \mid \xi_k)\, p^{\det}_{\Xi_k|\mathrm{B}_1,...,\mathrm{B}_k,\Upsilon_1,...,\Upsilon_{k-1},\Lambda_1,...,\Lambda_{k-1}}(\xi_k).$$

Substituting (14) and the pdf of the detected-target process with intensity $f_{k|k-1}^{Detected}(x_k)$ gives

$$\begin{aligned} &p_{\Upsilon_k|\mathrm{B}_1,...,\mathrm{B}_k,\Upsilon_1,...,\Upsilon_{k-1},\Lambda_1,...,\Lambda_{k-1}}(\upsilon_k) \\ &= \int_{\mathcal{S}} \cdots \int_{\mathcal{S}} \left(\prod_{j=1}^{m} p_{Z_k|X_k}\left(z_j \mid x_j^{\xi_k}\right) \right) \frac{e^{-\int_{\mathcal{S}} f_{k|k-1}^{Detected}(x)dx}}{m!} \prod_{j=1}^{m} f_{k|k-1}^{Detected}\left(x_j^{\xi_k}\right) dx_j^{\xi_k} \\ &= \frac{e^{-\int_{\mathcal{S}} P_k^D(x) f_{k|k-1}(x)dx}}{m!} \prod_{j=1}^{m} \int_{\mathcal{S}} p_{Z_k|X_k}\left(z_j \mid x_j^{\xi_k}\right) P_k^D\left(x_j^{\xi_k}\right) f_{k|k-1}\left(x_j^{\xi_k}\right) dx_j^{\xi_k}. \end{aligned}$$

Define

$$\lambda_{k|k-1}^{Target}(z_k) = \int_{\mathcal{S}} p_{Z_k|X_k}(z_k \mid x_k) P_k^D(x_k)\, f_{k|k-1}(x_k)\, dx_k, \qquad (15)$$

for $z_k \in \mathcal{Z}$. Since

$$\begin{aligned} &\int_{\mathcal{Z}} \lambda_{k|k-1}^{Target}(z_k)\, dz_k \\ &= \int_{\mathcal{S}} \left(\int_{\mathcal{Z}} p_{Z_k|X_k}(z_k \mid x_k)\, dz_k \right) P_k^D(x_k)\, f_{k|k-1}(x_k)\, dx_k \\ &= \int_{\mathcal{S}} P_k^D(x_k)\, f_{k|k-1}(x_k)\, dx_k = \int_{\mathcal{S}} f_{k|k-1}^{Detected}(x_k)\, dx_k, \end{aligned} \qquad (16)$$

the predicted target-originated measurement variate (11) is Poisson and $\lambda_{k|k-1}^{Target}(z_k)$ is its intensity.

4.2 Predicted total measurement intensity

The predicted total measurement variate including clutter contributions at time t_k is

$$\Upsilon_k \mid \mathrm{B}_1,...,\mathrm{B}_k,\Upsilon_1,...,\Upsilon_{k-1},\Lambda_1,...,\Lambda_k\,. \qquad (17)$$

The predicted measurement variate (17) is defined to be the superposition of the clutter-originated measurement process Λ_k and the target-originated measurement process (11). The predicted total measurement variate (17) is Poisson, and

$$\begin{aligned} \lambda_{k|k-1}(z_k) &= \lambda_k^{Clutter}(z_k) + \lambda_{k|k-1}^{Target}(z_k) \\ &= \lambda_k^{Clutter}(z_k) + \int_{\mathcal{S}} p_{Z_k|X_k}(z_k \mid x_k) P_k^D(x_k)\, f_{k|k-1}(x_k)\, dx_k \end{aligned} \qquad (18)$$

is its intensity.

5 Updated total target intensity

The updated total target intensity is the superposition of two processes. One is an information updated detected-target variate that is extended to a Poisson process on $\mathscr{E}(\mathcal{S})$. The other is the undetected-target Poisson process. The detected-target process is discussed first.

5.1 Information updated detected-target process

Let $Sym(m)$ denote the set of all permutations on the integers $\{1,...,m\}$, and let

$$\upsilon_k = \left(m^{\upsilon_k}, z_1^{\upsilon_k}, ..., z_{m^{\upsilon_k}}^{\upsilon_k}\right) \in \mathscr{E}(\mathcal{Z}) \qquad (19)$$

denote the given measurement at time t_k. When no clutter is present, there is a one-to-one correspondence between measurement data points and detected targets.

Appendix 2 shows that given there are m targets detected, the posterior process of the undetected targets is a Poisson point process with intensity function $\left(1 - P_k^D(s)\right) f_{k/k-1}(s)$ independent of the number and states of the detected targets. Thus the posterior state process of detected targets is independent of the process of undetected targets. This allows us to compute the posterior of these two processes independently.

Since there are m^{υ_k} data points in υ_k, there must be m^{υ_k} targets detected, and the multitarget measurement likelihood function is defined by

$$p_{\Upsilon_k|\Xi_k}\left(\upsilon_k \mid (x_1,\ldots,x_m)\right)= \begin{cases} \dfrac{1}{m!}\displaystyle\sum_{\sigma\in Sym(m)}\prod_{j=1}^{m} p_{Z_k|X_k}\left(z_j^{\upsilon_k} \mid x_{\sigma(j)}\right), & \text{if } m=m^{\upsilon_k}, \\ 0, & \text{if } m\neq m^{\upsilon_k}. \end{cases} \quad (20)$$

The summand in (20) is the measurement likelihood function conditioned on the hypothesis that the permuted states $\left(x_{\sigma(1)},\ldots,x_{\sigma(m)}\right)$ are the correct assignments for the ordered data $\left(z_1^{\upsilon_k},\ldots,z_m^{\upsilon_k}\right)$. Since the order of the given data set is uninformative, all permutations are *a priori* equally likely to be the correct assignment, so that $\Pr[\sigma]=1/m!$ The likelihood function (20) is clearly incorrect if the data contains clutter or false alarms. The problem of incorporating clutter into the model is discussed below in Section 4.4.

The information update for the detected-target process is

$$p_{\Xi_k|\mathrm{B}_1,\ldots,\mathrm{B}_k,\Upsilon_1,\ldots,\Upsilon_k,\Lambda_1,\ldots,\Lambda_k}(\xi_k) = p_{\Upsilon_k|\Xi_k}(\upsilon_k \mid \xi_k)\frac{p_{\Xi_k|\mathrm{B}_1,\ldots,\mathrm{B}_k,\Upsilon_1,\ldots,\Upsilon_{k-1},\Lambda_1,\ldots,\Lambda_k}(\xi_k)}{p_{\Upsilon_k|\mathrm{B}_1,\ldots,\mathrm{B}_k,\Upsilon_1,\ldots,\Upsilon_{k-1},\Lambda_1,\ldots,\Lambda_k}(\upsilon_k)}. \quad (21)$$

Since there are no clutter-originated data in this case, $\lambda_{k|k-1}(z)=\lambda_{k|k-1}^{Target}(z)$. For $m^{\upsilon_k}=m^{\xi_k}\equiv m$, using (20), the pdfs of the Poisson variates (9) and (17), and the identity (16) gives

$$\begin{aligned} &p_{\Xi_k|\mathrm{B}_1,\ldots,\mathrm{B}_k,\Upsilon_1,\ldots,\Upsilon_k,\Lambda_1,\ldots,\Lambda_k}(\xi_k) \\ &= \frac{1}{m!}\sum_{\sigma\in Sym(m)}\left\{\prod_{j=1}^{m} p_{Z_k|X_k}\left(z_j^{\upsilon_k} \mid x_{\sigma(j)}^{\xi_k}\right)\right\} \times \left\{\frac{\dfrac{e^{-\int f_{k|k-1}^{Detected}(x)dx}}{m!}\displaystyle\prod_{j=1}^{m} f_{k|k-1}^{Detected}\left(x_j^{\xi_k}\right)}{\dfrac{e^{-\int \lambda_{k|k-1}^{Target}(z)dz}}{m!}\displaystyle\prod_{j=1}^{m}\lambda_{k|k-1}^{Target}\left(z_j^{\upsilon_k}\right)}\right\} \\ &= \frac{1}{m!}\sum_{\sigma\in Sym(m)}\prod_{j=1}^{m}\frac{p_{Z_k|X_k}\left(z_{\sigma^{-1}(j)}^{\upsilon_k} \mid x_j^{\xi_k}\right)P_k^D\left(x_j^{\xi_k}\right)f_{k|k-1}\left(x_j^{\xi_k}\right)}{\lambda_{k|k-1}^{Target}\left(z_{\sigma^{-1}(j)}^{\upsilon_k}\right)}. \end{aligned} \quad (22)$$

The intensity (12) is substituted in the last step. For $m^{\upsilon_k}\neq m^{\xi_k}$, the information update is zero. The marginal on $x_s^{\xi_k}$, $s=1,\ldots,m$, is

$$p_{X_s^{\xi_k}|\mathrm{B}_1,\ldots,\mathrm{B}_k,\Upsilon_1,\ldots,\Upsilon_k,\Lambda_1,\ldots,\Lambda_k}\left(x_s^{\xi_k}\right)= \int_{\mathcal{S}}\cdots\int_{\mathcal{S}} p_{\Xi_k|\mathrm{B}_1,\ldots,\mathrm{B}_k,\Upsilon_1,\ldots,\Upsilon_k,\Lambda_1,\ldots,\Lambda_k}(\xi_k)\prod_{\substack{i=1\\ i\neq s}}^{m} dx_i^{\xi_k}. \quad (23)$$

Substituting (22) into (23) gives, using (15),

$$\begin{aligned} &p_{X_s^{\xi_k}|\mathrm{B}_1,\ldots,\mathrm{B}_k,\Upsilon_1,\ldots,\Upsilon_k,\Lambda_1,\ldots,\Lambda_k}\left(x_s^{\xi_k}\right) \\ &= \frac{1}{m!}\sum_{\sigma\in Sym(m)}\int_{\mathcal{S}}\cdots\int_{\mathcal{S}} \prod_{j=1}^{m}\frac{p_{Z_k|X_k}\left(z_{\sigma^{-1}(j)}^{\upsilon_k} \mid x_j^{\xi_k}\right)P_k^D\left(x_j^{\xi_k}\right)f_{k|k-1}\left(x_j^{\xi_k}\right)}{\lambda_{k|k-1}^{Target}\left(z_{\sigma^{-1}(j)}^{\upsilon_k}\right)}\prod_{\substack{i=1\\ i\neq s}}^{m} dx_i^{\xi_k} \\ &= \frac{1}{m!}\sum_{r=1}^{m}\sum_{\substack{\sigma\in Sym(m)\\ \text{such that}\\ \sigma^{-1}(s)=r}}\frac{p_{Z_k|X_k}\left(z_r^{\upsilon_k} \mid x_s^{\xi_k}\right)P_k^D\left(x_s^{\xi_k}\right)f_{k|k-1}\left(x_s^{\xi_k}\right)}{\lambda_{k|k-1}^{Target}\left(z_r^{\upsilon_k}\right)} \\ &= \frac{1}{m}\sum_{r=1}^{m}\frac{p_{Z_k|X_k}\left(z_r^{\upsilon_k} \mid x_s^{\xi_k}\right)P_k^D\left(x_s^{\xi_k}\right)f_{k|k-1}\left(x_s^{\xi_k}\right)}{\lambda_{k|k-1}^{Target}\left(z_r^{\upsilon_k}\right)}. \end{aligned} \quad (24)$$

The marginal pdfs are identical for all s. Let $p_{X_k|\mathrm{B}_1,\ldots,\mathrm{B}_k,\Upsilon_1,\ldots,\Upsilon_k,\Lambda_1,\ldots,\Lambda_k}(x_k)$ denote the marginal pdf.

5.2 Poisson approximation to information updated detected-target process

The variate $\Xi_k \mid \mathrm{B}_1,\ldots,\mathrm{B}_k,\Upsilon_1,\ldots,\Upsilon_k,\Lambda_1,\ldots,\Lambda_k$ is nonzero only for $m^{\upsilon_k}=m^{\xi_k}\equiv m$. Its pdf (22) is symmetric, but it does not necessarily factor in the manner necessary for it to be the pdf of a Poisson variate. The Poisson approximation is defined by the factorization

$$p_{\Xi_k|\mathrm{B}_1,\ldots,\mathrm{B}_k,\Upsilon_1,\ldots,\Upsilon_k,\Lambda_1,\ldots,\Lambda_k}\left(m,x_1^{\xi_k},\ldots,x_m^{\xi_k}\right) \cong \prod_{s=1}^{m} p_{X_k|\mathrm{B}_1,\ldots,\mathrm{B}_k,\Upsilon_1,\ldots,\Upsilon_k,\Lambda_1,\ldots,\Lambda_k}\left(x_s^{\xi_k}\right), \quad (25)$$

where the right hand side is interpreted as the pdf of the multitarget state $\left(x_1^{\xi_k},\ldots,x_m^{\xi_k}\right)$ conditioned on the number of measurements m. For any constant $c>0$, the intensity

$$f_{k|k}^{Detected}(x_k)\equiv c\; p_{X_k|\mathrm{B}_1,\ldots,\mathrm{B}_k,\Upsilon_1,\ldots,\Upsilon_k,\Lambda_1,\ldots,\Lambda_k}(x_k) \quad (26)$$

defines a Poisson process that satisfies the approximation (25). The maximum likelihood (ML) estimate of c is found from the likelihood of $\xi_k=\left(m,x_1^{\xi_k},\ldots,x_m^{\xi_k}\right)$. Using an obvious shorthand notation for the pdf in (26) gives the likelihood in the form

$$L(c \mid \xi_k)=\frac{\exp\left(-\int c p_{X_k|\cdots}(x)\,dx\right)}{m!}\prod_{s=1}^{m}\left(c p_{X_k|\cdots}\left(x_s^{\xi_k}\right)\right) \propto e^{-c}c^m. \quad (27)$$

It follows from (27) that $\hat{c}_{ML}=m$ for all $\left(x_1^{\xi_k},\ldots,x_m^{\xi_k}\right)$. Substituting $\hat{c}_{ML}$ and (24) into (26) gives

$$\begin{aligned} f_{k|k}^{Detected}(x_k) &= m\; p_{X_k|\mathrm{B}_1,\ldots,\mathrm{B}_k,\Upsilon_1,\ldots,\Upsilon_k,\Lambda_1,\ldots,\Lambda_k}(x_k) \\ &= \sum_{j=1}^{m}\frac{p_{Z_k|X_k}\left(z_j^{\upsilon_k} \mid x_k\right)P_k^D(x_k)\,f_{k|k-1}(x_k)}{\lambda_{k|k-1}^{Target}\left(z_j^{\upsilon_k}\right)}. \end{aligned} \quad (28)$$

The information updated detected-target Poisson process at time t_k is defined on the full event space $\mathscr{E}(\mathcal{S})$ via the intensity (28).

5.3 Updated total target intensity without clutter-originated data

The updated target process is the superposition of the detected-target and undetected-target processes. The undetected-target process is a Poisson process with intensity

$$f_{k|k}^{Undetected}(x_k) = \left(1 - P_k^D(x_k)\right) f_{k|k-1}(x_k), \quad (29)$$

so that the updated total target intensity at time t_k is, using (28) and (29),

$$\begin{aligned} f_{k|k}(x_k) &= f_{k|k}^{Undetected}(x_k) + f_{k|k}^{Detected}(x_k) \\ &= \left[1 - P_k^D(x_k) + \sum_{j=1}^{m} \frac{p_{Z_k|X_k}\left(z_j^{\upsilon_k} \mid x_k\right) P_k^D(x_k)}{\lambda_{k|k-1}^{Target}\left(z_j^{\upsilon_k}\right)}\right] f_{k|k-1}(x_k). \end{aligned} \quad (30)$$

Consequently, for all $\xi_k = \left(m^{\xi_k}, x_1^{\xi_k}, \ldots, x_{m^{\xi_k}}^{\xi_k}\right) \in \mathscr{E}(\mathcal{S})$,

$$p_{\Xi_k|\mathrm{B}_1,\ldots,\mathrm{B}_k,\Upsilon_1,\ldots,\Upsilon_k,\Lambda_1,\ldots,\Lambda_k}(\xi_k) = \frac{e^{-\int f_{k|k}(x)dx}}{m^{\xi_k}!} \prod_{j=1}^{m^{\xi_k}} f_{k|k}\left(x_j^{\xi_k}\right) \quad (31)$$

is the pdf of the total target Poisson variate $\Xi_k \mid \mathrm{B}_1,\ldots,\mathrm{B}_k,\Upsilon_1,\ldots,\Upsilon_k,\Lambda_1,\ldots,\Lambda_k$ when there is no clutter-originated data.

5.4 Clutter state model for clutter-originated data

The absence of clutter-originated data does not seem to be easily overcome while retaining an exact Bayesian information update of the multitarget pdf. The problems begin with (22) because the exponential terms do not cancel, and continue with (24) because the integrals do not integrate to one. Corrective multiplicative factors are easily found, and these factors are small when $\lambda_k^{Clutter}(z_k) \ll \lambda_k^{Target}(z_k)$; however, these problems arise because the likelihood function (20) assumes that all data points in the measurement set (19) are target-originated, not the superposition of target- and clutter-originated data. When clutter-originated data are included, the likelihood function (20) is no longer correct.

The approach taken is to augment the target space $\mathcal{S}$ with a clutter space $\mathcal{S}_\phi$. Defining the concept of target intensity on the augmented space $\mathcal{S} \cup \mathcal{S}_\phi$ requires the defining concept of target intensity on $\mathcal{S}_\phi$. The interpretation is that a target anywhere in $\mathcal{S}_\phi$ is a "clutter-target." The one measurement per target rule is enforced on the augmented space $\mathcal{S} \cup \mathcal{S}_\phi$, so that a clutter-target accounts for exactly one data point. Multiple clutter-targets are allowed. The likelihood function (20) extends to $\mathcal{S} \cup \mathcal{S}_\phi$ by interpreting the likelihood function $p_{Z_k|X_k}(z \mid x \in \mathcal{S}_\phi)$ to be the likelihood of the data under the "clutter origin" hypothesis. The sum over permutations in (20) is now an enumeration is over all possible assignments of data to either target or clutter-target. Consequently, evaluating the likelihood function (20) does not require knowing which data are clutter-target originated and which target-originated. In summary, on the augmented state space $\mathcal{S} \cup \mathcal{S}_\phi$, the likelihood function (20) is valid and the information update (30) holds exactly.

The augmented state space $\mathcal{S} \cup \mathcal{S}_\phi$ model leads to the idea of estimating the clutter intensity from the data, not specifying it *a priori*. The estimated clutter intensity (see (44) below) differs from the clutter-originated intensity $\lambda_k^{Clutter}(z_k)$ used in (18) because it is generated from the data via the target model, and is not an independent superposed measurement process. Replacing the estimated clutter intensity with the *a priori* known clutter intensity $\lambda_k^{Clutter}(z_k)$ seems reasonable in practice. This gives the PHD filter.

The clutter state can be taken to be an abstract point with nonzero intensity; however, this requires reworking earlier mathematical derivations. To avoid this, the clutter space is assumed to be a bounded subset of $\mathbb{R}^n$ with volume by $\varepsilon > 0$ and such that $\mathcal{S} \cap \mathcal{S}_\phi = \varnothing$. The augmented target state space $\mathcal{S} \cup \mathcal{S}_\phi$ is a disconnected subset of $\mathbb{R}^n$. The target intensity and filter are defined on $\mathcal{S} \cup \mathcal{S}_\phi$. This requires extending all relevant variables to $\mathcal{S} \cup \mathcal{S}_\phi$; for example, the transition probability function $\Psi_{X_k|X_{k-1}}(x \mid y)$ is extended to a function $\Psi_{X_k|X_{k-1}}^{Ext}(x \mid y)$ defined for all x and y in $\mathcal{S} \cup \mathcal{S}_\phi$. These extensions are chosen so that the updated extended total target intensity, denoted $f_{k|k}^{Ext}(x_k)$, is constant on $\mathcal{S}_\phi$ if $f_{k-1|k-1}^{Ext}(x_k)$ is constant on $\mathcal{S}_\phi$. The numerical value of the target intensity on $\mathcal{S}_\phi$ depends on its volume, ε.

The predicted total target intensity (10) and the predicted measurement intensity (15) are essentially unchanged, although the integrals are now over $\mathcal{S} \cup \mathcal{S}_\phi$. Let the inductively known target intensity on the augmented state space, $f_{k-1|k-1}^{Ext}(x_k)$ be such that $f_{k-1|k-1}^{Ext}(x_k) \equiv c_0$ for all $x_k \in \mathcal{S}_\phi$, where $c_0 \geq 0$ is a constant. Define

$$b_k^{Ext}(x_k) = \begin{cases} b_k(x_k), & \text{if } x_k \in \mathcal{S}, \\ \bar{b}_k, & \text{if } x_k \in \mathcal{S}_\phi, \end{cases} \quad (32)$$

and

$$d_k^{Ext}(x_k)=\begin{cases} d_k(x_k), & \text{if } x_k\in\mathcal{S}, \\ \bar{d}_k, & \text{if } x_k\in\mathcal{S}_\phi, \end{cases} \quad (33)$$

where the constants $\bar{b}_k$ and $\bar{d}_k$ satisfy $\bar{b}_k\ge 0$ and $0\le \bar{d}_k\le 1$. Define

$$\Psi^{Ext}_{X_k|X_{k-1}}(x_k\mid x_{k-1})=\begin{cases} c_1, & \text{for all } x_k\in\mathcal{S} \text{ and } x_{k-1}\in\mathcal{S}_\phi, \\ c_2, & \text{for all } x_k\in\mathcal{S}_\phi \text{ and } x_{k-1}\in\mathcal{S}_\phi, \end{cases} \quad (34)$$

and

$$\Psi^{Ext}_{X_k|X_{k-1}}(x_k\mid x_{k-1}) = \begin{cases} c_3\Psi_{X_k|X_{k-1}}(x_k\mid x_{k-1}), & \text{for all } x_k\in\mathcal{S} \text{ and } x_{k-1}\in\mathcal{S}, \\ c_4, & \text{for all } x_k\in\mathcal{S}_\phi \text{ and } x_{k-1}\in\mathcal{S}, \end{cases} \quad (35)$$

where $c_i\ge 0,\ i=1,\ldots,4$, are constants such that

$$\int_{\mathcal{S}\cup\mathcal{S}_\phi}\Psi^{Ext}_{X_k|X_{k-1}}(x_k\mid x_{k-1})\,dx_k=1, \quad \text{for all } x_{k-1}\in\mathcal{S}\cup\mathcal{S}_\phi. \quad (36)$$

Finally, define

$$p^{Ext}_{Z_k|X_k}(z_k\mid x_k)=\begin{cases} p_{Z_k|X_k}(z_k\mid x_k), & \text{if } x_k\in\mathcal{S}, \\ p_{Z_k|X_k}(z_k\mid\phi), & \text{if } x_k\in\mathcal{S}_\phi, \end{cases} \quad (37)$$

where $p_{Z_k|X_k}(z_k\mid\phi)$ is the likelihood function of the data under the "clutter origin" hypothesis.

The clutter state affects the multitarget likelihood function (20). For example, suppose there are $m=2$ measurements, $z_1^{\upsilon_k}$ and $z_2^{\upsilon_k}$. From the lower branch of (20) it follows that the only multitarget realizations with nonzero likelihood function values have two targets, $x_1^{\xi_k}$ and $x_2^{\xi_k}$, both of which are in the augmented state space $\mathcal{S}\cup\mathcal{S}_\phi$. Let $\upsilon_k=(2,z_1^{\upsilon_k},z_2^{\upsilon_k})$ and $\xi_k=(2,x_1^{\xi_k},x_2^{\xi_k})$. The measurement likelihood (20) on the augmented state space is then

$$\begin{aligned} p^{Ext}_{\Upsilon_k|\Xi_k}&(\upsilon_k\mid\xi_k) \\ &=\frac{1}{2}\sum_{\sigma\in Sym(2)} p^{Ext}_{Z_k|X_k}\left(z_1^{\upsilon_k}\mid x^{\xi_k}_{\sigma(j)}\right)p^{Ext}_{Z_k|X_k}\left(z_2^{\upsilon_k}\mid x^{\xi_k}_{\sigma(j)}\right) \quad (38) \\ &=\frac{1}{2}\left[\begin{aligned} &p^{Ext}_{Z_k|X_k}\left(z_1^{\upsilon_k}\mid x_1^{\xi_k}\right)p^{Ext}_{Z_k|X_k}\left(z_2^{\upsilon_k}\mid x_2^{\xi_k}\right) \\ &\quad + p^{Ext}_{Z_k|X_k}\left(z_1^{\upsilon_k}\mid x_2^{\xi_k}\right)p^{Ext}_{Z_k|X_k}\left(z_2^{\upsilon_k}\mid x_1^{\xi_k}\right)\end{aligned}\right]. \end{aligned}$$

There are four possibilities:

$$\begin{aligned} &\text{Case 1:} \quad x_1^{\xi_k}\in\mathcal{S} \text{ and } x_2^{\xi_k}\in\mathcal{S}, \\ &\text{Case 2:} \quad x_1^{\xi_k}\in\mathcal{S} \text{ and } x_2^{\xi_k}\in\mathcal{S}_\phi, \\ &\text{Case 3:} \quad x_1^{\xi_k}\in\mathcal{S}_\phi \text{ and } x_2^{\xi_k}\in\mathcal{S}, \\ &\text{Case 4:} \quad x_1^{\xi_k}\in\mathcal{S}_\phi \text{ and } x_2^{\xi_k}\in\mathcal{S}_\phi. \end{aligned} \quad (39)$$

Using (37), the measurement likelihood functions (38) for these four cases are

$$p^{Ext}_{\Upsilon_k|\Xi_k}(\upsilon_k\mid\xi_k) = \begin{cases} \frac{1}{2}\left[\begin{aligned} &p_{Z_k|X_k}\left(z_1^{\upsilon_k}\mid x_1^{\xi_k}\right)p_{Z_k|X_k}\left(z_2^{\upsilon_k}\mid x_2^{\xi_k}\right) \\ &\quad + p_{Z_k|X_k}\left(z_1^{\upsilon_k}\mid x_2^{\xi_k}\right)p_{Z_k|X_k}\left(z_2^{\upsilon_k}\mid x_1^{\xi_k}\right)\end{aligned}\right], & \text{Case 1,} \\ \frac{1}{2}\left[\begin{aligned} &p_{Z_k|X_k}\left(z_1^{\upsilon_k}\mid x_1^{\xi_k}\right)p_{Z_k|X_k}\left(z_2^{\upsilon_k}\mid \phi\right) \\ &\quad + p_{Z_k|X_k}\left(z_1^{\upsilon_k}\mid \phi\right)p_{Z_k|X_k}\left(z_2^{\upsilon_k}\mid x_1^{\xi_k}\right)\end{aligned}\right], & \text{Case 2,} \\ \frac{1}{2}\left[\begin{aligned} &p_{Z_k|X_k}\left(z_1^{\upsilon_k}\mid \phi\right)p_{Z_k|X_k}\left(z_2^{\upsilon_k}\mid x_2^{\xi_k}\right) \\ &\quad + p_{Z_k|X_k}\left(z_1^{\upsilon_k}\mid x_2^{\xi_k}\right)p_{Z_k|X_k}\left(z_2^{\upsilon_k}\mid \phi\right)\end{aligned}\right], & \text{Case 3,} \\ p_{Z_k|X_k}\left(z_1^{\upsilon_k}\mid\phi\right)p_{Z_k|X_k}\left(z_2^{\upsilon_k}\mid\phi\right), & \text{Case 4.} \end{cases}$$

All the various ways in which the data can arise from either targets or clutter are accommodated by the augmented state.

The predicted total target intensity (10) is, for $x_k\in\mathcal{S}\cup\mathcal{S}_\phi$,

$$\begin{aligned} f^{Ext}_{k|k-1}(x_k)&=b_k(x_k)+\hat{b}_k(x_k)+ \\ &\int_{\mathcal{S}}\Psi^{Ext}_{X_k|X_{k-1}}(x_k\mid x_{k-1})\left(1-d_{k-1}(x_{k-1})\right)f^{Ext}_{k-1|k-1}(x_{k-1})\,dx_{k-1}, \end{aligned} \quad (40)$$

where

$$\hat{b}_k(x_k)\equiv\int_{\mathcal{S}_\phi}\Psi^{Ext}_{X_k|X_{k-1}}(x_k\mid x_{k-1})\left(1-\bar{d}_{k-1}\right)f^{Ext}_{k-1|k-1}(x_{k-1})\,dx_{k-1} \quad (41)$$

is the estimated birth rate. From (32)-(36) and the inductive hypothesis that $f^{Ext}_{k-1|k-1}(x_{k-1})=c_0$ for all $x_k\in\mathcal{S}_\phi$, it follows that all three terms in (40) are constants for $x_k\in\mathcal{S}_\phi$. Hence, $f^{Ext}_{k|k-1}(x_k)$ is constant for $x_k\in\mathcal{S}_\phi$.

Define the extended probability of target detection as

$$P_k^{D/Ext}(x_k)=\begin{cases} P_k^D(x_k), & \text{if } x_k\in\mathcal{S}, \\ \bar{P}_k^D, & \text{if } x_k\in\mathcal{S}_\phi, \end{cases} \quad (42)$$

where the probability $\bar{P}_k^D\ge 0$ is a constant. The probability of detection parameter for the clutter is incorporated into the predicted intensity by setting $\bar{P}_k^D=1$. The predicted total measurement intensity (15) becomes, for $z_k\in\mathcal{Z}$,

$$\begin{aligned} \lambda^{Ext}_{k|k-1}(z_k)&=\int_{\mathcal{S}\cup\mathcal{S}_\phi}p^{Ext}_{Z_k|X_k}(z_k\mid x_k)P_k^{D/Ext}(x_k)\,f^{Ext}_{k|k-1}(x_k)\,dx_k \\ &=\hat{\lambda}^{Clutter}_{k|k-1}(z_k)+\int_{\mathcal{S}}p_{Z_k|X_k}(z_k\mid x_k)P_k^D(x_k)\,f^{Ext}_{k|k-1}(x_k)\,dx_k, \end{aligned} \quad (43)$$

where the estimated clutter intensity is

$$\begin{aligned} \hat{\lambda}^{Clutter}_{k|k-1}(z_k)&=p_{Z_k|X_k}(z_k\mid\phi)\int_{\mathcal{S}_\phi}f^{Ext}_{k|k-1}(x_k)\,dx_k \\ &\equiv p_{Z_k|X_k}(z_k\mid\phi)N^{Ext}_{k|k-1}(\phi), \end{aligned} \quad (44)$$

where $N_{k|k-1}^{Ext}(\phi)$ is the predicted number of targets in the clutter space. The estimated clutter target intensity fluctuates because it accounts for data originated by targets in the clutter space.

The total target intensity update (30) holds on the augmented state space $\mathcal{S} \cup \mathcal{S}_\phi$. For $x_k \in \mathcal{S}$ the updated intensity is

$$f_{k|k}^{Ext}(x_k) = f_{k|k-1}^{Ext}(x_k) \times \left[1 - P_k^D(x_k) + \sum_{j=1}^{m} \frac{p_{Z_k|X_k}(z_j^{\upsilon_k} \mid x_k) P_k^D(x_k)}{\hat{\lambda}_{k|k-1}^{Clutter}(z_j^{\upsilon_k}) + \int_{\mathcal{S}} p_{Z_k|X_k}(z_j^{\upsilon_k} \mid x_k) P_k^D(x_k) f_{k|k-1}^{Ext}(x_k) dx_k} \right] \quad (45)$$

and for $x_k \in \mathcal{S}_\phi$ it is

$$f_{k|k}^{Ext}(x_k) = f_{k|k-1}^{Ext}(x_k) \times \left[\sum_{j=1}^{m} \frac{p_{Z_k|X_k}(z_j^{\upsilon_k} \mid \phi)}{\hat{\lambda}_{k|k-1}^{Clutter}(z_j^{\upsilon_k}) + \int_{\mathcal{S}} p_{Z_k|X_k}(z_j^{\upsilon_k} \mid x_k) P_k^D(x_k) f_{k|k-1}^{Ext}(x_k) dx_k} \right]. \quad (46)$$

From (46) it follows that $f_{k|k}^{Ext}(x_k)$ is constant for all $x_k \in \mathcal{S}_\phi$. Multiplying both sides of (46) by ε gives

$$N_{k|k}^{Ext}(\phi) = N_{k|k-1}^{Ext}(\phi) \times \left[\sum_{j=1}^{m} \frac{p_{Z_k|X_k}(z_j^{\upsilon_k} \mid \phi)}{\hat{\lambda}_{k|k-1}^{Clutter}(z_j^{\upsilon_k}) + \int_{\mathcal{S}} p_{Z_k|X_k}(z_j^{\upsilon_k} \mid x_k) P_k^D(x_k) f_{k|k-1}^{Ext}(x_k) dx_k} \right], \quad (47)$$

where $N_{k|k}^{Ext}(\phi)$ is the information updated expected number of targets in $\mathcal{S}_\phi$.

The PHD filter [1] is recovered by setting $\hat{b}_k(x_k) \equiv 0$ in (40), $\hat{\lambda}_{k|k-1}^{Clutter}(z_k) \equiv \lambda_k^{Clutter}(z_k)$ in (45), and restricting the filter to the space $\mathcal{S}$. This is a reasonable procedure when the specified target birth process and clutter process intensities are accurately known.

Target birth and death models may make target transitions into and out $\mathcal{S}_\phi$ redundant. These transitions can be eliminated by setting $c_1 = c_4 = 0$ in (34) and (35). For this specialized transition function, the predicted target intensity (40) simplifies. The target birth correction term (41) vanishes for $x_k \in \mathcal{S}$, so that

$$f_{k|k-1}^{Ext}(x_k) = b_k(x_k) + \int_{\mathcal{S}} \Psi_{X_k|X_{k-1}}(x_k \mid x_{k-1}) \left(1 - d_{k-1}(x_{k-1})\right) f_{k-1|k-1}^{Ext}(x_{k-1}) dx_{k-1}. \quad (48)$$

For $x_k \in \mathcal{S}_\phi$, the integral in (40) vanishes but the correction term (41) does not. Multiplying by ε gives

$$N_{k|k-1}^{Ext}(\phi) = \bar{b}_k + \left(1 - \bar{d}_{k-1}\right) N_{k-1|k-1}^{Ext}(\phi). \quad (49)$$

The updated total target intensity is unchanged from (45) and (47).

6 Concluding remarks

The inclusion of the clutter-originated measurement process Λ_k leads to a complicated form of the information updated detected-target marginal pdf. The addition of a clutter state $\mathcal{S} \cup \mathcal{S}_\phi$ to the original target state space $\mathcal{S}$ remedies this technical difficulty and yields a tractable information updated detected-target marginal pdf on the augmented state space $\mathcal{S} \cup \mathcal{S}_\phi$. In effect, targets in $\mathcal{S}_\phi$ correspond to clutter-originated data. On the augmented state space, the Bayes net for the detected-target pdf does not have the clutter node Λ_k.

The information updated detected-target pdf is approximated by the product of its single-target marginal densities. The information updated target process is defined to be the sum of the undetected target Poisson process and the Poisson process obtained from the approximate detected-target pdf factorization.

References

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Appendix 1. Bayes net

The conditioning of the Poisson variates is expressed as a Bayesian inference net in Figure 1. The Bayes net for the augmented state space $\mathcal{S} \cup \mathcal{S}_\phi$ does not include the clutter node. Target birth and clutter birth variables are analogous to control terms in Bayesian single target formulations.

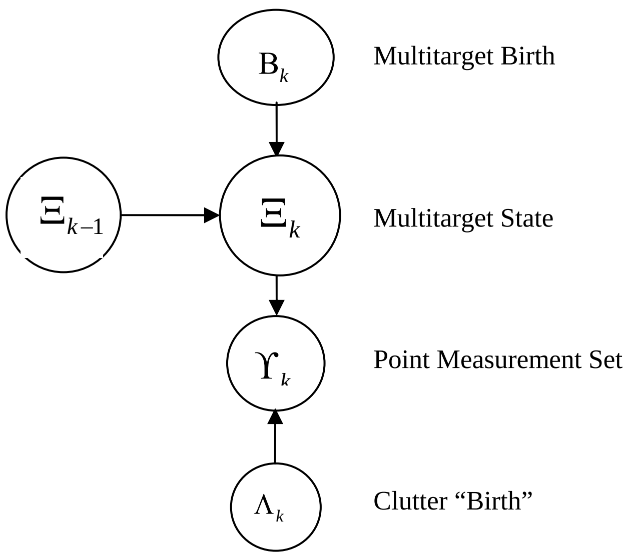

Figure 1. Bayes net for multitarget intensity filters

Appendix 2. Bernoulli filtered Poisson point processes

Assume that each target present at time t_k has an independent opportunity of dying with probability $d_k(s)$ which depends target state. If the target state process at time t_k is a Poisson point process, then the process of targets that remain alive is also a Poisson point process.

To see this, suppose ξ_k is a Poisson point process with target intensity f_k. Let $\mu_k \equiv \int_S f_k(s)ds$, $\mu_k(d) \equiv \int_S d(s) f_k(s)ds$, and $\delta_k = \mu_k(d)/\mu_k$. Let M_k be the number of targets in ξ_k and N_k be the number of targets alive after the death process takes place. Then $\Pr\{N_k = n \mid M_k = m\} = \binom{m}{n} \delta_k^{m-n} (1-\delta_k)^n$ for $n \le m$ and, for $n = 0,1,\ldots$,

$$\Pr\{N_k = n\} = \sum_{m=n}^{\infty} \binom{m}{n} \delta_k^{m-n} (1-\delta_k)^n \Pr\{\mathrm{M_k} = m\}$$

$$= \sum_{m=n}^{\infty} \frac{m!}{n!(m-n)!} \delta_k^{m-n} (1-\delta_k)^n \frac{(\mu_k)^m}{m!} e^{-\Lambda_k}$$

$$= \frac{((1-\delta_k)\mu_k)^n}{n!} e^{-\mu_k} \sum_{m=n}^{\infty} \frac{(\delta_k \mu_k)^{m-n}}{(m-n)!}$$

$$= \frac{((1-\delta_k)\mu_k)^n}{n!} e^{-\mu_k} e^{\delta_k \mu_k} = \frac{((1-\delta_k)\mu_k)^n}{n!} e^{-(1-\delta_k)\mu_k},$$

which is a Poisson distribution with mean

$$(1-\delta_k)\mu_k = \int_S (1-d_k(s)) f_k(s)ds .$$

A simple Bayesian posterior computation shows that the targets that survive have state distributions that are independent draws from the density $(1-d_k(s)) f_k(s)/(1-\delta_k)\mu_k$. Thus the point process of the surviving targets is a Poisson point process with intensity function $(1-d_k(s)) f_k(s)$ for $s \in \mathcal{S}$.

A similar argument shows that the multitarget state processes of the detected and undetected targets are Poisson point processes with intensity functions $P_k^D(s) f_{k/k-1}(s)$ and $(1-P_k^D(s)) f_{k/k-1}(s)$, respectively.

Appendix 3. Markov transformed Poisson point processes

Assume that each target moves according to the Markovian motion model : for $x_k, x_{k-1} \in S$, $k = 1,\ldots$

$$\Psi_k(x_k \mid x_{k-1}) = \Pr\{X_k = x_k \mid X_{k-1} = x_{k-1}\}. \qquad (50)$$

Under this model a Poisson point process at t_{k-1} with target intensity $f_{k-1}(\cdot)$ is transformed into a Poisson point process at time t_k with target intensity function

$$f_k(x) = \int_S \Psi_k(x \mid s) f_{k-1}(s)ds . \qquad (51)$$

To see this, let ξ_k be the target state process at time t_k for $k = 0,1,\ldots$, and $\mu_{k-1} \equiv \int_S f_{k-1}(s)ds$. Assume that ξ_{k-1} is a Poisson point process with target intensity function $f_{k-1}(\cdot)$. Then

$$\Pr\{\xi_k = (m, x_1, \ldots, x_m)\}$$

$$= \int_S \cdots \int_S \prod_{j=1}^m \Psi_k(x_j \mid s_j) \Pr\{\xi_{k-1} = (m, s_1, \ldots, s_m)\} ds_1 \ldots ds_m$$

$$= \frac{e^{-\mu_{k-1}}}{m!} \int_S \cdots \int_S \prod_{j=1}^m \Psi_k(x_j \mid s_j) \prod_{j=1}^m f_{k-1}(s_j) ds_1 \ldots ds_m$$

$$= \frac{e^{-\mu_{k-1}}}{m!} \prod_{j=1}^m \int_S \Psi_k(x_j \mid s_j) f_{k-1}(s_j) ds_j .$$

Since,

$$\int_S \int_S \Psi_k(x_j \mid s_j) f_{k-1}(s_j) ds_j dx_j = \int_S f_{k-1}(s_j) ds_j = \mu_{k-1},$$

it follows that ξ_k is a Poisson point process with target density function f_k as given by (51).