

CONSTRAINED OPTIMIZATION OF FUNCTIONALS WITH SEARCH THEORY APPLICATIONS*†

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We find necessary and sufficient conditions for maximizing a wide class of nonlinear, nonseparable functionals under separable constraints. The crucial restriction on the functionals is that they have a Gateaux differential which is a linear functional with a kernel. The conditions obtained can be applied to a large variety of optimal search problems involving moving targets when effort is infinitely divisible in space. Moreover, the conditions have been used to construct very efficient algorithms for solving these problems. It is conjectured that these results are useful in a general class of optimization problems that extend well beyond the search theory examples presented in this paper.

1. Introduction. We consider a functional P defined on a space Ψ of functions which satisfy certain linear, separable constraints. We find a set of conditions which are necessary and sufficient for maximizing P over Ψ . For the necessity of the conditions, the crucial requirement is that P have a Gateaux differential which is a linear functional defined by integration with a kernel function. For sufficiency we add the assumption that P is concave.

The results in this paper are a culmination of a number of papers which have found either necessary or necessary and sufficient conditions for maximizing probability of detection in optimal search problems. Hellman [4] found necessary conditions for targets moving according to a diffusion process and Saretsalo [7] generalized them to a large class of Markov processes. Pursiheimo [6] and Stone [10] found necessary and sufficient conditions for target motions of a special class called conditionally deterministic. Brown [1] found necessary and sufficient conditions for discrete time and space target motions and produced the first efficient algorithm for calculating optimal search plans for mixtures of discrete time and space Markov chains. Washburn [14] generalized Brown's necessary conditions to the case of discrete search effort. All of the above results, with the exception of those of Pursiheimo [6], Stone [10], and Washburn [14] assume an exponential detection function. Stone [11] found necessary and sufficient conditions for concave detection functions with essentially no restrictions on the stochastic process used to model target motion. With the exception of Washburn [14], all of the above results are special cases of the last result.

The conditions obtained in this paper are a generalization of those obtained in [11]. In addition to being more general, the necessity proof given in Theorem 1 of this paper is much simpler than that in [11] because it does not require the use of the Dubovitskii-Milyutin theorem or a measurable selection argument involving function spaces. For Theorem 2, we prove sufficiency by use of a lemma which allows one to obtain a bound on how far a given function in Ψ is from the optimal one. This bound, a generalization and tightening of a bound given by Washburn in [15], is very useful in

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providing efficient stopping criteria for algorithms based on the necessary and sufficient conditions.

Our results apply to a wide class of problems which includes many interesting problems from search theory. Some examples of these problems are given in §4. The examples are all related to search problems, but we conjecture that there are a number of interesting problems not directly related to search to which the conditions of this paper apply.

The results of this paper apply to search theory problems in which effort can be distributed as finely as desired over the search space and the effort allocation at one time does not restrict that at any other time. Results for the optimal searcher path problem, in which both of the above assumptions are violated, are much more meager. See Lukka [5] and Stewart [8] for discussions of this problem.

It is worth emphasizing that our results say nothing about the existence or uniqueness of functions satisfying the necessary and sufficient conditions. In fact, L. K. Arnold has constructed examples of search problems in which no allocation function satisfies the necessary condition. (In these examples, there do seem to be optimal search plans, but they require concentration of effort in sets of measure zero and therefore cannot be described exactly by allocation functions.) There are many examples in which optimal functions exist but are not unique.

In spite of the usefulness of our results, at least for search theory problems, in developing algorithms for finding optimal functions in Ψ , these results do not, to our knowledge, follow from standard optimization techniques, such as control theory, nonlinear programming or dynamic programming.

2. Necessary conditions. Let Y and T be spaces with σ -finite measures ν and τ , respectively. Let μ be either the product measure on $Y \times T$, or (if ν and τ are complete) the completion of the product measure. Let Z be a μ -measurable subset of $Y \times T$, and denote its T -sections by $Z_t = \{y \in Y | (y, t) \in Z\}$.

Let $c : Z \rightarrow (0, \infty)$ be μ -measurable, and let $m : T \rightarrow [0, \infty)$ be τ -measurable. Let Ψ be the set of μ -measurable functions $\psi : Z \rightarrow [0, \infty)$ such that

$$\int_{Z_t} c(y, t)\psi(y, t)d\nu(y) \leq m(t) \tag{1}$$

for a.e. $t \in T$, and let Ψ_0 be the set of $\psi \in \Psi$ such that

$$\int_{Z_t} c(y, t)\psi(y, t)d\nu(y) = m(t)$$

for a.e. $t \in T$.

In the examples presented in §4, a function ψ represents an allocation of effort over space Y and time T . The cost of allocating a unit of effort to y at time t is $c(y, t)$, and (1) represents a constraint on total cost at each time. The set of allocations satisfying the constraint is Ψ , and Ψ_0 is the set of allocations for which the constraint is binding at almost all times. In most applications, units of measurement can be chosen which make $c(y, t)$ identically 1.

For each $\psi \in \Psi$, let $K(\psi)$ be the set of μ -measurable functions $h : Z \rightarrow (-\infty, \infty)$ such that $(\psi + \epsilon h) \in \Psi$ for all sufficiently small positive ϵ .

Let P be a real-valued functional on Ψ . If $\psi \in \Psi$ and $h \in K(\psi)$, define the *Gateaux Differential* of P at ψ in the direction of h by

$$P'[\psi, h] = \lim_{\epsilon \rightarrow 0^+} \frac{1}{\epsilon} (P[\psi + \epsilon h] - P[\psi]) \tag{2}$$

if the limit exists. Suppose that there is a μ -measurable function $d(\psi, \cdot, \cdot) : Z \rightarrow (-\infty,$

∞) such that for every $h \in K(\psi)$, the Gateaux Differential exists and is given by

$$P'[\psi, h] = \int_{\mathbf{Z}} d(\psi, y, t)h(y, t)d\mu(y, t). \tag{3}$$

Then $d(\psi, \cdot, \cdot)$ is a *kernel* of the Gateaux Differential at ψ . This possibly odd-looking requirement will be motivated by the examples in §4. We list ψ as an argument of d in order to emphasize that the kernel may be different or may fail to exist for other choices of ψ . Note that $d(\psi, y, t)$ does not mean $d(\psi(y, t), y, t)$.

For technical convenience we have used a one-sided limit in (2), although in the usual definition of Gateaux Differential, the limit is two sided. Note that for functionals satisfying (3), the limit in (2) is actually two sided whenever $P[\psi + \epsilon h]$ is defined for small negative ϵ .

We say that $\psi^* \in \Psi$ is *optimal* (or *optimizes* $P[\psi^*]$ *within* Ψ) if $P[\psi^*] \geq P[\psi]$ for all $\psi \in \Psi$. The following theorem provides a necessary condition for ψ^* to be optimal.

THEOREM 1. *Let Ψ be (as above) the set of measurable functions $\psi: \mathbf{Z} \rightarrow [0, \infty)$ satisfying (1), let $\psi^* \in \Psi$, and let P be a real-valued functional on Ψ . Assume that P has a Gateaux differential at ψ^* with kernel $d(\psi^*, \cdot, \cdot)$. Then a necessary condition for ψ^* to be optimal is that there exists a measurable function $\lambda: \mathbf{T} \rightarrow [-\infty, \infty]$ such that for a.e. (y, t) .*

$$\begin{aligned} d(\psi^*, y, t) &\leq \lambda(t)c(y, t), \quad \text{and} \\ d(\psi^*, y, t) &= \lambda(t)c(y, t), \quad \text{if } \psi^*(y, t) > 0. \end{aligned} \tag{4}$$

PROOF. Let ψ^* be optimal. For a.e. $t \in \mathbf{T}$, define

$$\begin{aligned} u(t) &= \text{ess sup}_{y \in \mathbf{Z}_t} \frac{d(\psi^*, y, t)}{c(y, t)}, \quad \text{and} \\ w(t) &= \text{ess inf}_{\{y \in \mathbf{Z}_t : \psi^*(y, t) > 0\}} \frac{d(\psi^*, y, t)}{c(y, t)}. \end{aligned} \tag{5}$$

(Recall that by definition $\text{ess sup}_{y \in A} f(y)$ is the smallest r such that $f(y)$ exceeds r only on a set of measure zero in A . The definition of ess inf is similar.) Lemma 1 below implies that both u and w are τ -measurable functions from T into $[-\infty, \infty]$.

Let $S = \{t \mid u(t) > w(t)\}$. Suppose first that $\tau(S) = 0$; then $u(t) \leq w(t)$ for almost every $t \in T$. We can now set $\lambda(t) = u(t)$, and equations (5) imply equations (4) for almost every (y, t) , proving the theorem in this case. (There is a trivial special case, in which the infimum in (5) is taken over a set of measure zero. This is the only case in which $u(t) < w(t)$ is possible, or in which $\lambda(t)$ may equal $\pm \infty$; but even in this case, the theorem holds as written.)

Now suppose that $\tau(S) > 0$. In this case we will construct a function $\psi \in \Psi$ for which $P[\psi] > P[\psi^*]$, thus contradicting the optimality of ψ^* . First let $v: S \rightarrow (-\infty, \infty)$ be any τ -measurable function such that

$$w(t) < v(t) < u(t) \quad \text{for } t \in S. \tag{6}$$

(For example we may let

$$\begin{aligned} v(t) &= \frac{1}{2}(u(t) + w(t)) && \text{if both } u(t) \text{ and } w(t) \text{ are finite;} \\ &= w(t) + 1 && \text{if } u(t) = +\infty, w(t) \text{ finite;} \\ &= u(t) \div 1 && \text{if } u(t) \text{ finite, } w(t) = -\infty; \text{ and} \\ &= 0 && \text{if } u(t) = +\infty, w(t) = -\infty.) \end{aligned}$$

Now define subsets of \mathbf{Z} by

$$A = \left\{ (y, t) \mid t \in S \text{ and } \frac{d(\psi^*, y, t)}{c(y, t)} > v(t) \right\}, \text{ and}$$

$$B = \left\{ (y, t) \mid t \in S, \psi^*(y, t) > 0, \text{ and } \frac{d(\psi^*, y, t)}{c(y, t)} < v(t) \right\}.$$

Both A and B are μ -measurable sets. We denote their \mathbf{T} -sections by $A_t = A \cap \mathbf{Z}_t$ and $B_t = B \cap \mathbf{Z}_t$. For every $t \in S$, both $\nu(A_t)$ and $\nu(B_t)$ are positive by (5) and (6). Hence,

$$\mu(A) = \int_S \nu(A_t) d\tau(t) \quad \text{and} \quad \mu(B) = \int_S \nu(B_t) d\tau(t)$$

are both positive.

Let $a: \mathbf{Y} \rightarrow (0, 1]$ be a measurable function whose integral over \mathbf{Y} is 1; that is, $\int_{\mathbf{Y}} a(y) d\nu(y) = 1$. Such a function exists because \mathbf{Y} is σ -finite.

Now define a function $h: \mathbf{Z} \rightarrow (-\infty, \infty)$ by

$$h(y, t) = -\psi^*(y, t)a(y) \quad \text{if } (y, t) \in B;$$

$$h(y, t) = \frac{a(y)}{c(y, t)} k(t) \quad \text{if } (y, t) \in A; \text{ and}$$

$$h(y, t) = 0 \quad \text{for all other } (y, t),$$

where $k(t)$ (contrived for the sake of equation (7), below) is given by

$$k(t) = \frac{\int_{B_t} c(z, t)\psi^*(z, t)a(z) d\nu(z)}{\int_{A_t} a(z) d\nu(z)}.$$

To see that h is measurable, first note that A , B , ψ^* , and a are measurable, so the only problem would be with the second formula. Both integrals in the definition of $k(t)$ are measurable functions of t by Fubini's Theorem. Our construction guarantees that the integral in the denominator is positive (because $\nu(A_t) > 0$) and finite (because $\int_{A_t} a(z) d\nu(z) \leq \int_{\mathbf{Y}} a(z) d\nu(z) = 1$). Therefore, both k and h are measurable. The finiteness of h now follows from that of the numerator in the definition of $k(t)$,

$$\int_{B_t} c(z, t)\psi^*(z, t)a(z) d\nu(z) \leq \int_{\mathbf{Z}_t} c(z, t)\psi^*(z, t) d\nu(z) \leq m(t) < \infty.$$

Now notice that $(\psi^* + \epsilon h)$ is nonnegative for $0 \leq \epsilon \leq 1$, and that for a.e. $t \in \mathbf{T}$,

$$\int_{\mathbf{Z}_t} c(y, t)h(y, t) d\nu(y) = 0. \tag{7}$$

Therefore, $(\psi^* + \epsilon h) \in \Psi$ for small positive ϵ , and $h \in K(\psi^*)$.

Now compute $P'[\psi^*, h]$:

$$P'[\psi^*, h] = \int_{\mathbf{Z}} d(\psi^*, y, t)h(y, t) d\mu(y, t)$$

$$= \int_{\mathbf{Z}} (d(\psi^*, y, t) - v(t)c(y, t))h(y, t) d\mu(y, t), \tag{8}$$

where the last line follows from (7). The last integrand in (8) is zero unless $(y, t) \in A$ or $(y, t) \in B$. If $(y, t) \in A$, then

$$d(\psi^*, y, t) - v(t)c(y, t) > 0 \quad \text{and} \quad h(y, t) > 0;$$

and if $(y, t) \in B$, then

$$d(\psi^*, y, t) - v(t)c(y, t) < 0 \quad \text{and} \quad h(y, t) < 0.$$

In either case, the integrand is positive and both cases occur on sets of positive measure. Therefore,

$$P'[\psi^*, h] > 0.$$

It now follows from the definition of the Gateaux Differential that for sufficiently small $\epsilon > 0$,

$$P[\psi^* + \epsilon h] > P[\psi^*],$$

which contradicts the optimality of ψ^* .

This completes the proof of the theorem, except for the proof of Lemma 1. This proof is routine but it is included here because it is not easily found in the literature.

LEMMA 1. *Let \mathbf{Y} and \mathbf{T} be spaces with σ -finite measures ν and τ , respectively, and let μ be either the product measure or (if ν and τ are complete) the completion of the product measure on $\mathbf{Y} \times \mathbf{T}$. Let $g: \mathbf{Y} \times \mathbf{T} \rightarrow [-\infty, \infty]$ be μ -measurable and define $f: \mathbf{T} \rightarrow [-\infty, \infty]$ by*

$$f(t) = \text{ess sup}_{y \in \mathbf{Y}} g(y, t).$$

Then f is τ -measurable.

PROOF. Let $a \in \mathbf{R}$, the reals, and let $A = \{t \mid f(t) \leq a\}$. It will suffice to show that A is measurable.

Write $B = \{(y, t) \mid g(y, t) > a\}$; then B is a μ -measurable set. By applying Fubini's Theorem to its characteristic function, we conclude that $B_t = \{y \mid (y, t) \in B\}$ is measurable for all t . (In the "complete case," i.e., if μ is the completion of the product measure, B_t is measurable for a.e. t .) In any case, $\nu(B_t)$ is a measurable function of t .

By definition,

$$f(t) = \inf \{r \in \mathbf{R} \mid \nu\{y \mid g(y, t) > r\} = 0\}.$$

(By contention $\inf \mathbf{R} = -\infty$ and \inf (empty set) = $+\infty$.) It follows that $f(t) \leq a$ if and only if $B_t = \{y \mid g(y, t) > a\}$ has ν -measure zero. Therefore,

$$A = \{t \mid f(t) \leq a\} = \{t \mid \nu(B_t) = 0\}$$

(in the complete case, this equality holds except for a set of τ -measure zero). Since $\nu(B_t)$ is a measurable function of t , this implies that A is measurable.

3. Sufficient conditions and upper bounds. We now examine a situation in which the necessary condition in Theorem 1 is also sufficient.

Let Ψ and Ψ_0 be as before, and recall that $\psi \in \Psi_0$ implies that

$$\int_{\mathbf{Z}_t} c(y, t)\psi(y, t) d\mu(y, t) = m(t) \quad \text{for a.e. } t \in \mathbf{T}.$$

Note that Ψ is convex. The functional P is concave on Ψ if, for all $\psi, \psi' \in \Psi$, and all $0 \leq \epsilon \leq 1$, we have

$$P[(1 - \epsilon)\psi + \epsilon\psi'] \geq (1 - \epsilon)P[\psi] + \epsilon P[\psi']. \tag{8}$$

LEMMA 2. *Let P be a concave functional on Ψ . Let $\psi \in \Psi_0$ and assume that P has a Gateaux Differential at ψ with kernel $d(\psi, \cdot, \cdot)$. Let*

$$\bar{\lambda}(\psi, t) = \text{ess sup}_{y \in \mathbf{Z}_t} \frac{d(\psi, y, t)}{c(y, t)}.$$

Then for any $\psi' \in \Psi$, we have

$$P[\psi'] \leq P[\psi] + \int_Z (\bar{\lambda}(\psi, t)c(y, t) - d(\psi, y, t))\psi(y, t)d\mu(y, t). \tag{9}$$

PROOF. First, using the concavity of P , we calculate:

$$\begin{aligned} P'[\psi, \psi' - \psi] &= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} (P[(1 - \epsilon)\psi + \epsilon\psi'] - P[\psi]) \\ &\geq \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} ((1 - \epsilon)P[\psi] + \epsilon P[\psi'] - P[\psi]) \\ &= \lim_{\epsilon \rightarrow 0} (P[\psi'] - P[\psi]) = P[\psi'] - P[\psi]. \end{aligned} \tag{10}$$

From the existence of the kernel, we obtain

$$\begin{aligned} P'[\psi, \psi' - \psi] &= \int_Z d(\psi, y, t)(\psi'(y, t) - \psi(y, t))d\mu(y, t) \\ &= \int_Z d(\psi, y, t)\psi'(y, t)d\mu(y, t) - \int_Z d(\psi, y, t)\psi(y, t)d\mu(y, t). \end{aligned} \tag{11}$$

From the definition of $\bar{\lambda}$, we have $d(\psi, y, t) \leq \bar{\lambda}(\psi, t)c(y, t)$, so that

$$\begin{aligned} \int_Z d(\psi, y, t)\psi'(y, t)d\mu(y, t) &\leq \int_Z \bar{\lambda}(\psi, t)c(y, t)\psi'(y, t)d\mu(y, t) \\ &= \int_T \bar{\lambda}(\psi, t) \int_{Z_t} c(y, t)\psi'(y, t)dv(y)d\tau(t) \\ &\leq \int_T \bar{\lambda}(\psi, t)m(t)d\tau(t) \\ &= \int_T \bar{\lambda}(\psi, t) \int_{Z_t} c(y, t)\psi(y, t)dv(t)d\tau(t) \\ &= \int_Z \bar{\lambda}(\psi, t)c(y, t)\psi(y, t)d\mu(y, t). \end{aligned}$$

Finally, combining this result with (10) and (11) yields

$$\begin{aligned} P[\psi'] - P[\psi] &\leq P'[\psi, \psi' - \psi] \\ &\leq \int_Z \bar{\lambda}(\psi, t)c(y, t)\psi(y, t)d\mu(y, t) - \int_Z d(\psi, y, t)\psi(y, t)d\mu(y, t) \\ &= \int_Z (\bar{\lambda}(\psi, t)c(y, t) - d(\psi, y, t))\psi(y, t)d\mu(y, t), \end{aligned}$$

which implies the conclusion of the lemma.

Note that the lemma establishes an upper bound for $P[\psi']$ which depends only on the behavior of P near ψ . This upper bound is similar to the one given by Washburn [15]. In fact if one defines

$$\underline{\lambda}(t) = \operatorname{ess\,inf}_{\{y \in Z_t : \psi(y, t) > 0\}} \frac{d(\psi, y, t)}{c(y, t)} \quad \text{for } t \in T,$$

then

$$P[\psi'] - P[\psi] \leq \int_T (\bar{\lambda}(t) - \underline{\lambda}(t))m(t)d\tau(t),$$

which is a direct generalization of Washburn's bound. The bound in Lemma 2 is sharper although potentially more difficult to compute.

Using Lemma 2 we prove the following theorem, which establishes a case in which the necessary conditions of Theorem 1 are also sufficient.

THEOREM 2. *In addition to the hypotheses of Theorem 1, assume that P is concave and that $\psi^* \in \Psi_0$. Then the necessary conditions of Theorem 1 are also sufficient for ψ^* to be optimal.*

PROOF. Let $\psi' \in \Psi$. By Lemma 2 we have

$$P[\psi'] \leq P[\psi^*] + \int_{\mathbf{Z}} (\bar{\lambda}(\psi^*, t)c(y, t) - d(\psi^*, y, t))\psi^*(y, t)d\mu(y, t), \tag{12}$$

where

$$\bar{\lambda}(\psi^*, t) = \text{ess sup}_{y \in \mathbf{Z}_t} \frac{d(\psi^*, y, t)}{c(y, t)}.$$

The necessary condition in Theorem 1 implies that $d(\psi^*, y, t) = \bar{\lambda}(\psi^*, t)c(y, t)$ whenever $\psi^*(y, t)$ is not zero, except on a set of measure zero in \mathbf{Z} ; and this implies that the integrand in (12) is zero almost everywhere in \mathbf{Z} . Therefore,

$$P[\psi'] \leq P[\psi^*],$$

completing the proof that ψ^* is optimal.

Note. Theorems 1 and 2, as well as both lemmas, can be proved without the requirement that $d(\psi^*, \cdot, \cdot)$ be μ -measurable; it is only necessary that the iterated integral

$$\int_{\mathbf{T}} \int_{\mathbf{Z}_t} d(\psi^*, y, t) d\nu(y) d\tau(t)$$

exist. Also, allocations ψ with a similar measurability property must be allowed. The proofs are the same, although some steps are more delicate; and they render unnecessary the assumption that \mathbf{T} is σ -finite.

4. Examples. In this section we present a number of examples which illustrate the types of functionals to which the above necessary and sufficient conditions can be applied. These examples are all related to search theory. This is a natural consequence of the background of the authors. However, we hope that the reader will be stimulated to supply examples from his own experience that fit into the general framework developed here.

Optimal detection search. Suppose that a target is moving through the plane \mathbf{Y} according to the discrete-time stochastic process $X = \{X_t : t = 0, 1, \dots, T\}$. We have $m(t)$ effort available at each time t and a function $\psi \in \Psi$ specifies $\psi(y, t)$, the search effort density placed at point y at time t for $y \in \mathbf{Y}$ and $t = 0, 1, \dots, T$. We take $\mathbf{T} = \{0, 1, \dots, T\}$, $\mathbf{Z} = \mathbf{Y} \times \mathbf{T}$, ν to be Lebesgue measure, and τ to be counting measure. We let $c(y, t) = 1$ for $y \in \mathbf{Y}$, $t \in \mathbf{T}$.

For each sample path of the process X , the probability of detecting the target given it follows that path is a function of the weighted total effort density

$$\zeta(\psi, T) = \sum_{t=0}^T W(X_t, t)\psi(X_t, t),$$

which accumulates on the target over the course of that path. The weight $W(y, t)$ represents the relative detectability or sweep width against the target if it is located at point y at time t . There is a function $b : [0, \infty) \rightarrow [0, 1]$, such that $b(\zeta(\psi, T))$ = probability of detecting the target by time T given it follows the given path and

effort density ψ is applied. Letting E denote expectation over the sample paths of X , we define

$$P_T[\psi] = E[b(\zeta(\psi, T))]$$

to be the probability of detecting the target by time T with plan ψ .

The optimal detection problem is to find a plan $\psi^* \in \Psi$ such that $P_T[\psi^*] \geq P_T[\psi]$ for all $\psi \in \Psi$. Such a plan is called *T-optimal*.

Let E_{y_t} denote expectation conditioned on $X_t = y$ and let p_t be the probability density function for X_t . If b has a bounded nonnegative derivative b' , then the Gateaux Differential of P_T exists and is given by (see Stone [11])

$$P'_T[\psi, h] = \int_Y \sum_{t=0}^T d(\psi, y, t) h(y, t) dy \quad \text{for } h \in K(\psi),$$

where

$$d(\psi, y, t) = E_{y_t}[b'(\zeta(\psi, T))]W(y, t)p_t(y), \quad y \in Y, t \in T.$$

Taking $P = P_T$ in Theorem 1, defining d as above, and recalling that $c(y, t) = 1$ for $y \in Y$ and $t \in T$, we see that conditions (4) become

$$\begin{aligned} d(\psi^*, y, t) &= \lambda(t) \quad \text{for } \psi^*(y, t) > 0 \\ &\leq \lambda(t) \quad \text{for } \psi^*(y, t) = 0, \end{aligned} \tag{13}$$

where $\lambda(t) \geq 0$ for $t = 0, 1, \dots, T$.

Thus the necessary and sufficient conditions for optimal search for a moving target (Stone [11]) are a special case of Theorems 1 and 2. (The continuous time results in Stone [11] may be obtained from Theorems 1 and 2 by taking $T = [0, T]$ and τ to be Lebesgue measure on $[0, T]$.)

Brown's [1] conditions are the special case of the above conditions that is obtained by taking the detection function to be $b(z) = 1 - e^{-z}$ for $z \geq 0$, $Y = \{1, 2, \dots, J\}$, and ν to be counting measure on $\{1, 2, \dots, J\}$. The conditions of Brown and Stone have been used to generate efficient algorithms for finding *T-optimal* search plans for moving targets (see Brown [1] and Stone et al. [13]). Washburn's [14] upper bound, which is generalized by Lemma 2, has proved very useful in efficiently terminating these algorithms; see Stone and Kadane [12] and Discenza and Stone [3].

A special case of a moving target problem is a stationary one. In particular, by taking $T = \{0\}$, there is only one time period and we have a stationary target problem. In this case,

$$d(\psi, y, t) = b'(W(y, 0)\psi(y, 0))W(y, 0)p_0(y),$$

and the necessary and sufficient conditions for optimal search for a stationary target (see Stone [9, §2.1]) also become a special case of Theorems 1 and 2.

Multistate target search. A generalization of the detection search described above is the multistate target search. In this case, the target's motion and state are represented by a stochastic process $(X, S) = \{(X_t, S_t): t = 0, 1, \dots, T\}$ where X_t = target's position at time t and S_t = target's state at time t . The target may change states as well as locations stochastically and the target's state may affect the target's motion and detectability. Specifically, suppose there are $k = 1, \dots, K$ states and that the sweep width W is a function of location, state, and time so that

$$\zeta(\psi, T) = \sum_{t=0}^T W(X_t, S_t)\psi(X_t, t)$$

and

$$P_T[\psi] = E[b(\xi(\psi, T))].$$

Observe that effort cannot be allocated to states, only to locations. Let $E_{y,k,t}$ indicate expectation conditioned on $X_t = y$ and $S_t = k$ and let $p_t(y, k) = \Pr\{X_t = y, S_t = k\}$. Assuming b has bounded derivative b' , one can show (Discenza and Stone [3]) that the functional P_T has a Gateaux Differential with kernel

$$d(\psi, y, t) = \sum_{k=1}^K E_{y,k,t}[b'(\xi(\psi, T))]W(y, k, t)p(y, k, t)$$

for $y \in Y, t = 0, 1, 2, \dots, T.$ (14)

Using the definition of d given in equation (14) and setting $c(y, t) = 1$ for $y \in Y, t \in T$, one can check that the necessary and sufficient conditions for optimal multistate target search obtained in Discenza and Stone [3] are also a special case of Theorems 1 and 2.

Two special cases of multistate target search are survivor search and defensive search. In survivor search, the target may be a person missing at sea or lost in the wilderness. The state represents the condition of the survivor, e.g., in a boat, in a life raft, in the water, or dead. By setting the sweep width equal to its appropriate value for each state and zero if the target is dead, the problem of finding a plan to maximize P_T becomes that of finding a plan to maximize the probability of detecting the target alive by time T .

For a defensive search, Brown [2], we are trying to detect a target before it launches an attack or performs some undesirable action. In this case, the state variable tells whether the attack has been launched or not. Once the attack is launched, the process remains in this trapping state and the sweep width is taken to be zero. Thus, maximizing P_T is equivalent to maximizing the probability of detecting the target by time T and before it launches an attack.

Minimizing mean time to complete a search. In the search problems described above, there is no premium for detecting the target early in the interval $[0, T]$ rather than late. In most searches, the sooner one is successful, the better. One way to reflect this fact is to plan the search to minimize the mean time to complete the search. As above, we have a time horizon or cutoff time T . (This horizon may be ∞ .) The search proceeds until the target is found or the time T is reached. When the target is found or the cutoff time reached, we say the search is *completed*.

For a plan $\psi \in \Psi$, and $t = 0, 1, \dots, T$ define

$$\xi(\psi, t) = \sum_{s=0}^t W(X_s, t)\psi(X_s, t)$$

and

$$P_t[\psi] = E[b(\xi(\psi, t))] = \text{probability of detection by time } t.$$

Let

$$M[\psi] = \sum_{t=1}^T t[P_t[\psi] - P_{t-1}[\psi]] + (T+1)(1 - P_T[\psi]) = \sum_{t=0}^T (1 - P_t[\psi])$$

= mean time to complete the search. (15)

Observe that $M[\psi] = \sum_{t=0}^T (1 - P_t[\psi])$ holds for $T = \infty$ even if the mean time is infinite.

Now let $P[\psi] = -M[\psi]$. Then

$$P[\psi] = - \sum_{t=0}^T E[1 - b(\xi(\psi, t))].$$

As before let $E_{y,u}$ indicate expectation conditioned on $X_u = y$, and let $p_u(y) = \Pr\{X_u = y\}$. If T is finite, then one may compute that

$$\begin{aligned} P'[\psi, h] &= \sum_{t=0}^T E[b'(\xi(\psi, t))] \sum_{u=0}^t W(X_u, u)h(X_u, u) \\ &= \sum_{t=0}^T \sum_{u=0}^t \int_Y E_{y,u}[b'(\xi(\psi, t))] p_u(y)W(y, u)h(y, u)dv(y) \\ &= \sum_{u=0}^T \int_Y \sum_{t=u}^T E_{y,u}[b'(\xi(\psi, t))] p_u(y)W(y, u)h(y, u)dv(y). \end{aligned}$$

Thus,

$$d(\psi, y, u) = p_u(y)W(y, u) \sum_{t=u}^T E_{y,u}[b'(\xi(\psi, t))] \quad \text{for } y \in Y, u = 0, 1, \dots, T.$$

With d defined as above, $c(y, t) = 1$ for $y \in Y, t = 0, 1, \dots, T$, and $b(z) = 1 - e^{-z}$ for $z \geq 0$, one can check that the necessary and sufficient conditions obtained in Chapter V of Stone et al. [13] for minimizing expected time to complete a search are also a special case of Theorems 1 and 2.

Maximizing expected return. Let Ω be a probabilistic sample space, such that drawing an element ω from Ω determines the target's path. Suppose that the reward or return for detecting a target is given by a function $R : \Omega \times T \rightarrow (-\infty, \infty)$ so that if we detect the target at time t on path ω , we receive a return $R(\omega, t)$. Our object is to maximize the expected return. If the search terminates at time T without detection, then we assume there is a return $Q(t)$ which may be positive or negative. The latter could reflect a penalty for failing to detect.

In typical cases the reward may depend only on the place and time of detection; that is, $R(\omega, t)$ may depend only on X_t (determined by ω) and t . However, very little simplification results from restricting our attention to such cases. With the present formulation we allow the reward to depend on other random variables, possibly correlated with the target motion, whose value may also be determined by the choice of ω .

Let

$$\xi(\psi, t) = \sum_{s=0}^t W(X_s, s)\psi(X_s, s), \quad b(\xi(\psi, -1)) = 0,$$

and define for $\psi \in \Psi$

$$P[\psi] = E \left[\sum_{t=0}^T R(\omega, t)[b(\xi(\psi, t)) - b(\xi(\psi, t-1))] + (1 - b(\xi(\psi, T)))Q(T) \right].$$

Let $R(\omega, T+1) = Q(T)$ for all $\omega \in \Omega$. Then

$$P[\psi] = E \left[\sum_{t=0}^T [R(\omega, t) - R(\omega, t+1)]b(\psi(t)) \right] + Q(T). \tag{16}$$

The Gateaux differential of P is

$$\begin{aligned} P'[\psi, h] &= \sum_{t=0}^T E \left[[R(\omega, t) - R(\omega, t+1)] b'(\xi(\psi, t)) \sum_{u=0}^t W(X_u, u) h(X_u, u) \right] \\ &= \sum_{u=0}^T \int_{\mathbf{Y}} \sum_{t=u}^T E_{y,u} [[R(\omega, t) - R(\omega, t+1)] b'(\xi(\psi, t))] p_u(y) W(y, u) h(y, u) dy, \end{aligned}$$

and

$$d(\psi, y, u) = \sum_{t=u}^T E_{y,u} [[R(\omega, t) - R(\omega, t+1)] b'(\xi(\psi, t))] p_u(y) W(y, u).$$

The necessary conditions of Theorem 1 now apply. In order to obtain the sufficient conditions of Theorem 2, it suffices to assume that $R(\omega, \cdot)$ is a nonincreasing function for all $\omega \in \Omega$ and that b is concave in order to guarantee the concavity of P .

Observe that if $R(\omega, t) = -t$ for $\omega \in \Omega$ and $Q(T) = -(T+1)$, then $P[\psi]$ defined by equation (16) is $-M[\psi]$, the negative of the expected time to complete the search defined in the previous example.

Constraint on total effort. In the detection search example given above, search effort was available at a given rate and effort could not be shifted from one time period to another. Instead of this assumption, suppose that a fixed amount of effort is available for the whole search and that this effort can be freely shifted among time periods. To obtain necessary and sufficient conditions for this situation let \mathbf{R}^2 be the plane in which the target moves, $\mathbf{Y} = \mathbf{R}^2 \times \{0, 1, \dots, T\}$, and $\mathbf{T} = \{0\}$. Thus points $y \in \mathbf{Y}$ are pairs (x, t) where $x \in \mathbf{R}^2$ and $t \in \{0, 1, \dots, T\}$. Let ν be the product of Lebesgue and counting measure and τ be counting measure. Let $c(y, 0) = 1$ for $y \in \mathbf{Y}$ and $m(0)$ be the total effort available. Then Ψ is the class of search allocations $\psi: \mathbf{R}^2 \times \{0, 1, \dots, T\} \rightarrow [0, \infty)$ such that

$$\sum_{t=0}^T \int_{\mathbf{R}^2} \psi(x, t) dx \leq m(0), \quad (17)$$

and Ψ_0 is the set of ψ for which equality holds in (17).

The Gateaux differential of P has the kernel

$$d(\psi, x, t) = E_{x,t} [b'(\xi(\psi, T))] W(x, t) p_t(x) \quad \text{for } x \in \mathbf{R}^2, t = 0, 1, \dots, T,$$

and if b is concave the necessary and sufficient conditions of Theorems 1 and 2 for $\psi^* \in \Psi_0$ to be optimal have the following form: There exists a $\lambda \geq 0$ such that

$$\begin{aligned} d(\psi^*, x, t) &= \lambda \quad \text{if } \psi(x, t) > 0, \\ &\leq \lambda \quad \text{if } \psi(x, t) = 0, \quad \text{for } x \in \mathbf{R}^2, t = 0, 1, \dots, T. \end{aligned}$$

The difference between these conditions and those of (13) is that there is a single λ for all time periods when the constraint is on total effort only.

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