J. Appl. Prob. **9**, 769–788 (1972)
Printed in Israel

CROSSING PROBLEMS FOR NON-CONSTANT THRESHOLDS AND CERTAIN NON-MARKOV PROCESSES

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Abstract

A result in Stone, Belkin, and Snyder ((1970) *J. Math. Anal. Appl.* **30**, 448-470) gave a method for finding the Laplace-Stieltjes transform of the distribution of certain non-negative, homogeneous, additive functionals of a Markov process with stationary transition measure. By considering certain two dimensional Markov processes and applying this result, a method is obtained for finding time above a threshold and first passage distributions for a one dimensional process either when (1) the process is Markovian and the threshold is possibly non-constant, or (2) the threshold is constant and the process is the indefinite integral of a Markov process. Specific process-threshold combinations are considered in several examples including the case of a linear threshold for a Wiener process and a for compound Poisson process with exponential (either one-sided or bilateral) after-jump distribution. In addition, the first passage distribution to a constant threshold is computed for an integrated Poisson sampling process.

FIRST PASSAGE PROBLEMS; MARKOV PROCESSES; NON-CONSTANT THRESHOLD CROSSING; DISTRIBUTIONS OF FUNCTIONALS OF MARKOV PROCESSES

1. Introduction and basic methodology

In Stone, Belkin and Snyder (1970) a method was obtained for finding the Laplace-Stieltjes transform of the distribution of certain non-negative, homogeneous, additive functionals of a Markov process with stationary transition measure. A specialization of this result then gave a method for finding the double transform of the distribution of time spent above a constant threshold level in a specified time interval by such a process. In the several examples considered, the transform of the first passage distribution was obtained as a by-product by taking an appropriate limit.

In this paper we first state in Theorem 1.2 a result which allows one to obtain the transform of the first passage distribution directly. Then by considering special two-dimensional Markov processes and applying the results of Stone, Belkin and Snyder (1970) and Theorem 1.2 we show how time above a threshold or first passage distributions can be obtained for a one-dimensional process either when

Received 24 September 1971.

(1) the process is Markovian but the threshold is non-constant, or (2) the threshold is constant and the process can be made Markovian by expanding the state space to two dimensions (in particular, such is the case when the process is the indefinite integral of a Markov process). Specific process-threshold combinations are considered in several examples including the case of a linear threshold for both the Wiener process and a compound Poisson process with a bilateral exponential after-jump distribution. The first passage distribution to a constant threshold is also computed for an integrated Poisson sampling process.

In general we let $\{X_t: t \geqq 0\}$ be a measurable Markov process with stationary transition probabilities defined on a probability space $(\Omega, \mathscr{F}, P)$ with state space $(S, \mathscr{S})$. Let R^n denote Euclidean n-space and $L: R^1 \to R^1$. We are interested in the distribution of

$$\tau = \inf\{t: t \geqq 0, X_t > L(t)\},$$

the first passage time across the threshold L, and the distribution of time the process spends above L in $[0, t]$.

Let B be the Banach space of real-valued bounded Borel measurable functions on S with the supremum norm and define the transition semigroup operator $T_t: B \to B$ by

$$T_t f(x) = \int_S p(t, x, dy) f(y),$$

where $p: [0, \infty) \times S \times \mathscr{S} \to [0, 1]$ is the transition function of $\{X_t\}$. Denote by $\tilde{B}$ the set of all $f \in B$ such that

$$\lim_{t \downarrow 0} T_t f(x) = f(x) \text{ for } x \in S.$$

The weak infinitesimal operator $\mathscr{A}$ is defined by

$$\mathscr{A} f = \lim_{t \downarrow 0} (T_t f - f)/t,$$

whenever the limit exists in the weak sense and is in $\tilde{B}$.

For $h \geqq 0$ in B define the homogeneous, additive functional

$$H(t) = \int_0^t h[X_s(\omega)] ds$$

and let

$$u(x) = \int_0^\infty e^{-\alpha t} E_x[e^{-\beta H(t)}] dt. \tag{1.1}$$

Then the special form of Theorem 2.2 in Stone, Belkin and Snyder (1970) which we shall use is Theorem 1.1.

Theorem 1.1. Let $\{X_t\}$ be a measurable Markov process with stationary transition probabilities and weak infinitesimal operator $\mathscr{A}$. If $hu \in \tilde{B}$, then u defined by (1.1) is the unique, bounded, Borel measurable solution of

$$(\alpha - \mathscr{A} + \beta h)u = 1 \tag{1.2}$$

for $\alpha > \beta$.

In the context of the constant level crossing problem, h is taken as the indicator function of (l, ∞), where l is the level and $h(l)$ may have to be redefined to assure the condition $hu \in \tilde{B}$.

With this definition of h, use was made several times in the examples in Stone, Belkin and Snyder (1970) of the fact that the transform v of the first passage distribution could be obtained from u using the relationship

$$\begin{aligned} v(x) &= \int_0^\infty e^{-\alpha t} P_x[\tau > t]dt \\ &= \lim_{\beta \to \infty} \int_0^\infty e^{-\alpha t} E_x[e^{-\beta H(t)}]dt \\ &= \lim_{\beta \to \infty} u(x). \end{aligned}$$

In Example 3 involving an integrated Poisson sampling process we will be concerned only with the first passage distribution, and in general it would be convenient to have a more direct method for obtaining v. An analogue of Theorem 1.1 is in fact available based on the following approach.

Suppose one defines the process $\{\hat{X}_t\}$ as the Markov process $\{X_t\}$ with extinction time $\zeta = \inf\{t: t \geqq 0, X_t \in \Gamma\}$, where $\Gamma \in \mathscr{S}$. Denote the corresponding semigroup operators by $\hat{T}_t$, the weak infinitesimal operator by $\hat{\mathscr{A}}$, etc. Then the following result holds.

Theorem 1.2. Let $\{\hat{X}_t\}$ be a measurable Markov process with stationary and stochastically continuous transition measure, a topological state space $(S, \mathscr{C}, \mathscr{S})$ and with extinction time $\zeta = \inf\{t: t \geqq 0, \hat{X}_t \in \Gamma\}$ where $\Gamma \in \mathscr{S}$. Let I be a function such that

$$I(x) = \begin{cases} 1 \text{ for } x \in S - \bar{\Gamma} \\ 0 \text{ for } x \in \operatorname{int}\Gamma \\ \lim_{t \downarrow 0} \hat{T}_t I(x) \text{ for } x \in \partial\Gamma. \end{cases}$$

Define

$$w(x) = \hat{R}_\alpha I(x) = E_x \int_0^\zeta e^{-\alpha t} I(\hat{X}_t)dt.$$

Then

$$(\alpha - \hat{\mathscr{A}})w = I, \tag{1.3}$$

and w is the unique bounded, Borel measurable solution of (1.3).

Proof. I is a bounded Borel measurable function, and the hypothesis of stochastic continuity guarantees the property that

$$\lim_{t \downarrow 0} \hat{T}_t I(x) = I(x) \text{ for all } x \in S.$$

The fact that w satisfies (1.3) is then a direct consequence of Theorem 1.7 in Dynkin (1965) (also stated as Theorem 2.1 in Stone, Belkin and Snyder (1970)) together with the fact that w is in the range of the resolvent operator. Also, if w^* satisfies

$$(\alpha - \hat{\mathscr{A}})w^* = I,$$

then by the result cited above I is in the domain of the resolvent and

$$w^* = (\alpha - \hat{\mathscr{A}})^{-1} I = \hat{R}_\alpha I.$$

Uniqueness follows and this completes the proof.

Several remarks are required about the application of Theorem 1.2 to computing first passage distributions. First, one observes that

$$w(x) = E_x \int_0^\zeta e^{-\alpha t} I(\hat{X}_t) dt = \int_0^\infty e^{-\alpha t} E_x[\chi_{\{\zeta \geqq t\}} I(\hat{X}_t)] dt,$$

where χ is the set indicator function. On the other hand,

$$v(x) = \int_0^\infty e^{-\alpha t} P_x[\zeta \geqq t].$$

It is seen, therefore, that in general $w(x) = v(x)$ need not hold for all $x \in S$. This will, however, be the case whenever either of the following two conditions hold:

(i) $I(x) = 1$ for $x \in \partial\Gamma$,

(ii) $E_x[\mu\{t: X_t \in \partial\Gamma\}] = 0$ for $x \in S - \Gamma$,

where μ denotes Lebesgue measure.

We now show howto apply Theorems 1.1 and 1.2 to problems involving non-constant thresholds. Define an auxiliary sample space $\Omega \times R^1$, and for $(\omega, s) \in \Omega \times R^1$ let

$$Z_t(\omega, s) = (X_t(\omega), t + s). \tag{1.4}$$

Then $\{Z_t: t \geqq 0\}$ is a measurable Markov process with stationary transition probabilities. Let h be the indicator function of the set $\{(x, t): x > L(t)\}$. Define

(1.5) $$H(t,(\omega,s)) = \int_0^t h(X_r(\omega), r+s)dr \text{ for } t \geqq 0, \text{ and } (\omega,s) \in \Omega \times R.$$

Then $H(t,(\omega,s))$ gives the amount of time the process $\{X_t\}$ spends above the shifted threshold $L^*(t) = L(t+s)$ in the interval $[0,t]$.

For each $\omega \in \Omega$ and $r \geqq 0$, let ω_r^+ be such that $X_t(\omega_r^+) = X_{t+r}(\omega)$. For $0 \leqq r \leqq t$, it is clear that

$$H(t,(\omega,s)) = H(r,(\omega,s)) + H(t-r,(\omega_r^+, s+r)).$$

Henceforth we write $H(t)$ for $H(t,(\omega,s))$. Define

(1.6) $$u(x,s) = \int_0^\infty e^{-\alpha t} E_{x,s}[e^{-\beta H(t)}]dt.$$

Then Theorem 1.1 applies to the two-dimensional process $\{Z_t\}$ with u defined by (1.6).

The second application of Theorems 1.1 and 1.2 in two dimensions involves consideration of an integrated process

$$Y_t = Y_0 + \int_0^t X_s ds,$$

where Y_0 is a constant. While the process Y_t is not itself Markov, the two-dimensional process $Z_t = (X_t, Y_t)$ is Markovian with stationary transition measure. To apply the theorems one need only observe that $Y_t > l$ is equivalent to $Z_t \in \Gamma = R^1 \times (l, \infty)$ and therefore that h should be taken as the indicator function of Γ (except possibly on $\partial\Gamma$).

2. Example 1: time above a linear threshold for a Wiener process

In this example we apply Theorem 1.1 to the Wiener process to find explicitly the density for $H(t)$ for a linear threshold when the process starts on or below the threshold. In addition, we find the density for $H(\infty) = \lim_{t\to\infty} H(t)$.

Let $\{X_t : t \geqq 0\}$ be the standard Wiener process with $E[X_t - X_s] = 0$ and $\operatorname{var}[X_t - X_s] = |t-s|$. For $b \geqq 0$ and $a \in R^1$, let $L(s) = a + bs$, and define h to be the indicator function of the set $\{(x,s) : x > L(s)\}$ except when $L(x) = s$, in which case we take $h(x,s) = \frac{1}{2}$. Then defining $H(t)$ by (1.5), we have that $u(x,s)$ defined by (1.6) gives the double transform of the time that the Wiener process spends above the shifted threshold $L^*(r) = L(r+s)$ in the interval $[0,t]$ given $X_0 = x$. Since $\{X_t\}$ is homogeneous in space, we may take $a = 0$. Let

$$F_x(t,\tau) = \Pr\{H(t) \leqq \tau \mid X_0 = x, s = 0\}$$

and

$$G_x(\tau) = \Pr\{H(\infty) \leqq \tau \mid X_0 = x,\ s = 0\}.$$

The two main results of this example are that

$$\frac{\partial F_0(t,\tau)}{\partial \tau} = \tfrac{1}{2}\left[\sqrt{\frac{2}{\pi\tau}}\,e^{-\frac{1}{2}b^2\tau} - b\,\mathrm{erfc}(b\sqrt{\tfrac{1}{2}\tau})\right]$$

(2.1)
$$\times\left[\sqrt{\frac{2}{\pi(t-\tau)}}\,e^{-\frac{1}{2}b^2(t-\tau)} + b\,\mathrm{erfc}(-b\sqrt{\tfrac{1}{2}(t-\tau)})\right] \quad \text{for } t \geqq \tau \geqq 0,$$

and that for $b \geqq 0$,

(2.2)
$$\frac{\partial G_0(\tau)}{\partial \tau} = b\left[\sqrt{\frac{2}{\pi\tau}}\,e^{-\frac{1}{2}b^2\tau} - b\,\mathrm{erfc}(b\sqrt{\tfrac{1}{2}\tau})\right] \quad \text{for } t \geqq 0.$$

Before proving (2.1) and (2.2) we pause to point out some interesting facts about these densities. Observe that if we take $b = 0$ in (2.1), then we obtain the classical arcsine density for the amount of time the Wiener process spends above 0 in the interval $[0,t]$. In this case, the highest likelihood (i.e., probability density) occurs when $\tau = 0$ and $\tau = t$, and the lowest likelihood occurs at $\tau = \frac{1}{2}t$. When $b > 0$, the highest likelihood still occurs at $\tau = 0$ and $\tau = t$. In addition, it is conjectured that the minimum likelihood occurs at $\tau > \frac{1}{2}t$ and that as $b \to \infty$, the minimum likelihood point approaches t.

In the case of $b = 0$, it is known that $H(\infty) = \infty$ with probability 1. However, when $b > 0$, one may integrate (2.2) to check that $H(\infty) < \infty$ with probability 1. In addition, the maximum likelihood for $H(\infty)$ occurs at $\tau = 0$. Observe also, that the density of $H(\infty)$ is proportional to the first factor of the density of $H(t)$ and that $\lim_{t\to\infty} \partial F_0(t,\tau)/\partial\tau = \partial G_0(\tau)/\partial\tau$.

In order to use Theorem 1.1 to find the double transform u, we compute the in finitesimal operator $\mathscr{A}$ for $\{Z_t\}$ defined by (1.4) when $\{X_t\}$ is a Wiener process by use of (4.2)–(4.4) on p. 321 of Feller (1966). Let f be a bounded Borel function $f: R^2 \to R^1$ in the domain of $\mathscr{A}$. We shall denote the partial derivative of f with respect to its ith and then jth variable by $f_{i,j}$. Then

$$\mathscr{A}f(x,s) = f_2(x,s) + \tfrac{1}{2}f_{1,1}(x,s).$$

One may check that u is a bounded continuous function. L is defined by $L(s) = bs$ (where $b > 0$). If $x = L(s)$, then

$$\lim_{t\to 0+} T_t h(x,s) = \lim_{t\to 0+} (2\pi t)^{-\frac{1}{2}} \int_{bt}^{\infty} e^{-y^2/2t}\,dy = \tfrac{1}{2}$$

so that $h \in \tilde{B}$. Thus by Corollary 2.2 of Stone, Belkin and Snyder (1970), u is the unique bounded Borel solution of (1.2), which becomes

(2.3)
$$(\alpha+\beta)u(x,s) - \tfrac{1}{2}u_{1,1}(x,s) - u_2(x,s) = 1 \qquad \text{for } x > bs,$$
$$\alpha u(x,s) - \tfrac{1}{2}u_{1,1}(x,s) - u_2(x,s) = 1 \qquad \text{for } x < bs.$$

Because the Wiener process is homogeneous in space, $u(x,s) = u(x - bs, 0)$ and the above equations become

$$\begin{aligned} &(\alpha+\beta)u(x-bs,0)-\tfrac{1}{2}u_{1,1}(x-bs,0)+bu_1(x-bs,0)=1 \qquad \text{for } x>bs,\\ (2.4)\qquad &\alpha u(x-bs,0)-\tfrac{1}{2}u_{1,1}(x-bs,0)+bu_1(x-bs,0)=1 \qquad \text{for } x<bs.\end{aligned}$$

Letting $y = x - bs$, (2.4) becomes

$$\begin{aligned} &(\alpha+\beta)u(y,0)-\tfrac{1}{2}u_{1,1}(y,0)+bu_1(y,0)=1 \qquad \text{for } y>0,\\ (2.5)\qquad &\alpha u(y,0)-\tfrac{1}{2}u_{1,1}(y,0)+bu_1(y,0)=1 \qquad \text{for } y<0.\end{aligned}$$

We solve (2.5) by obtaining a general solution, in the standard manner, to each of the equations in (2.5) in their respective domains. Each of these solutions contains two arbitrary constants. By imposing the boundary condition that $u(y,0)$ be bounded as $y \to \pm\infty$ in the first and second equations respectively, we are left with two arbitrary constants (one from each solution) to be determined. These are determined by matching the two solutions and their derivatives at $y = 0$. The result is that we obtain

$$\begin{aligned} (2.6)\qquad &(\alpha+\beta)u(y,0) = 1 + \frac{\beta}{\alpha}\,\frac{b+(b^2+2\alpha)^{\frac{1}{2}}}{(b^2+2\alpha)^{\frac{1}{2}}+(b^2+2(\alpha+\beta))^{\frac{1}{2}}}\\ &\qquad\times \exp[y(b-(b^2+2(\alpha+\beta))^{\frac{1}{2}})] \qquad \text{for } y>0,\\ &\alpha u(y,0) = 1 + \frac{\beta}{\alpha+\beta}\,\frac{b-(b^2+2(\alpha+\beta))^{\frac{1}{2}}}{(b^2+2\alpha)^{\frac{1}{2}}+(b^2+2(\alpha+\beta))^{\frac{1}{2}}}\\ &\qquad\times \exp[y(b+(b^2+2\alpha))^{\frac{1}{2}})] \qquad \text{for } y \leqq 0,\end{aligned}$$

where $u(0,0)$ is obtained by continuity of $u(\cdot\,,0)$ at $y = 0$.

To verify that (2.1) holds, one need only take the double transform of the right-hand side of (2.1) and observe that one obtains $u(0,0)$. A convenient way to do this is to transform first with respect to t and then τ making use of (29.2.15), (29.3.37) and (29.2.12) of Abramowitz and Stegun (1968).

In order to obtain (2.2) we consider

$$(2.7)\qquad \lim_{\alpha\to 0}\alpha u(0,0) = \frac{b}{\beta}[(b^2+2\beta)^{\frac{1}{2}}-b] = \frac{2b}{(b^2+2\beta)^{\frac{1}{2}}+b}.$$

From the definition of u given by (1.6), we find that the left-hand side of (2.7) is the transform of $\partial G_0(\tau)/\partial\tau$. Using (29.3.37) and (29.2.14) of Abramowitz and Stegun (1968) we find that (2.2) holds. From (2.7) one may also compute $E[H(\infty)] = \frac{1}{2}b^{-2}$.

In the case where $y < 0$, we may invert $u(y,0)$ to find that for $t \geqq \tau \geqq 0$,

$$\begin{aligned}\partial F_y(t,\tau)/\partial\tau &= \delta(\tau)P_y(t) + \tfrac{1}{2}\left\{\sqrt{\frac{2}{\pi\tau}}e^{-\frac{1}{2}b^2\tau} - b\,\mathrm{erfc}(b\sqrt{\tfrac{1}{2}\tau})\right\}\\ &\times\left\{\sqrt{\frac{2}{\pi(t-\tau)}}\exp\left[\frac{-[b(t-\tau)-y]^2}{2(t-\tau)}\right] + be^{2by}\,\mathrm{erfc}\left[\frac{-b(t-\tau)+y}{\sqrt{(2(t-\tau))}}\right]\right\}\end{aligned}$$

where δ is the Dirac delta function and

$$P_y(t) = 1 + y \int_0^t (2\pi s^3)^{-\frac{1}{2}} \exp[-(bs - y)^2/2s] ds,$$

which is just the probability that the process fails to cross the linear threshold $L(s) = bs$ in the interval $[0, t]$ given $X_0 = y$ (see Mehr and McFadden (1965)).

Note that for $y < 0$,

$$1 - \lim_{\beta \to \infty} \alpha u(y, 0) = \exp[y(b + (b^2 + 2\alpha)^{\frac{1}{2}})], \tag{2.8}$$

which is simply the transform of $-\partial P_y(t)/\partial t$, the density for the first passage time to L. By setting $\alpha = 0$ in (2.8), we find the well-known result that for $y < 0$, e^{2yb} is the probability that $\{X_t\}$ ever crosses L given $X_0 = y$. Thus we may write

$$\frac{\partial G_y(\tau)}{\partial \tau} = \delta(\tau)(1 - e^{2yb}) + e^{2yb} \frac{\partial G_0(\tau)}{\partial \tau}.$$

We are not able to perform the required inversions to obtain the distribution of $H(t)$ or $H(\infty)$ when the process starts above the threshold

3. Example 2: time above linear threshold for a compound Poisson process

In this example we find the double transform $\hat{u}$ defined by (1.6) for a Poisson process and a linear threshold in the case where the after-jump distribution is either a one-sided or two-sided negative exponential. In the one-sided case, we find an explicit expression (i.e., (3.8)) for the distribution of the amount of time the process spends above the threshold in the time interval $[0, \infty]$.

Let $\{U_n\}$, $n = 0, 1, \cdots$, be a sequence of independent identically distributed random variables, and define $U_0 = x$. Let $N(t)$ be the number of events which occur during $(0, t]$ in a Poisson process with intensity λ. $X_t = \sum_{n=0}^{N(t)} U_n$ is then a compound Poisson process. Let h be the indicator function of $\{(x, s): x > L(s)\}$ and let the threshold L be given by

$$L(s) = a + bs \text{ for } -\infty < s < \infty,$$

where $b > 0$. Because of the homogeneity in space of the compound Poisson process, it is sufficient to consider thresholds with $a = 0$.

Let f be a bounded Borel function such that $f: R^2 \to R^1$, and let f_2 denote the partial derivative of f with respect to its second variable. If f_2 exists and is bounded and $f_2(x, \cdot)$ is continuous for $x \in R$, then it is straightforward to check that

$$\mathscr{A} f(x, s) = \lambda \left[\int_R f(y, s) K(x, dy) - f(x, s) \right] + \frac{\partial f(x, s)}{\partial s},$$

where $K(x, A) = \Pr\{U_1 + x \in A\}$ for $A \in \mathscr{S}$.

One may use the increasing nature of L to check that $hu \in \tilde{B}$ so that we may apply Theorem 1.1 to conclude that u is the unique bounded Borel solution of

$$(\alpha+\beta+\lambda)u(x,s)-\lambda\int_R u(y,s)K(x,dy) - \frac{\partial u(x,s)}{\partial s} = 1, \qquad x > L(s),$$

$$(\alpha+\lambda)u(x,s)-\lambda\int_R u(y,s)K(x,dy) - \frac{\partial u(x,s)}{\partial s} = 1, \qquad x \leqq L(s).$$

Because the process is homogeneous in space, $u(y,s)$ depends only on $L(s)-y$. Since $bs-y = br-x$ if and only if $r = s-(y-x)/b$, we have $u(y,s) = u(x, s-(y-x)/b)$. This suggests the change of variable $z = (y-x)/b$ whereupon the above equations become

$$(3.1)\quad \begin{aligned}&(\alpha+\beta+\lambda)u(x,s)-\lambda b\int_R u(x,s-z)\phi(bz)dz - \frac{\partial u(x,s)}{\partial s} = 1, \qquad s < x/b,\\ &(\alpha+\lambda)u(x,s)-\lambda b\int_R u(x,s-z)\phi(bz)dz - \frac{\partial u(x,s)}{\partial s} = 1, \qquad s \geqq x/b,\end{aligned}$$

where K is assumed to have a density ϕ. By the above translation property it is clear that we need only find $u(0,s)$ for all $-\infty < s < \infty$ to determine u. We write $u(s)$ for $u(0,s)$.

To facilitate an application of Fourier transforms we modify the behavior of u at infinity by introducing the new functions

$$u^+(s) = \begin{cases} u(s)-(1/\alpha) & \text{for } s \geqq 0 \\ 0 & \text{for } s < 0, \end{cases}$$

$$u^-(s) = \begin{cases} 0 & \text{for } s \geqq 0 \\ u(s)-(1/(\alpha+\beta)) & \text{for } s < 0. \end{cases}$$

Equation (3.1) then becomes

$$(3.2)\quad \begin{aligned}&(\alpha+\beta+\lambda)u^-(s)+(\alpha+\lambda)u^+(s)-\lambda b\int_R [u^+(s-z)+u^-(s-z)]\phi(bz)dz - \frac{\partial u(s)}{\partial s}\\ &= -\frac{\lambda}{\alpha+\beta}[1-\xi(s)] - \frac{\lambda}{\alpha}\xi(s) + \lambda b\int_R\left[\frac{\lambda}{\alpha+\beta}[1-\xi(s-z)]+\frac{1}{\alpha}\xi(s-z)\right]\phi(bz)dz,\end{aligned}$$

where

$$\xi(s) = \begin{cases} 1 & \text{for } s > 0 \\ 0 & \text{for } s \leqq 0. \end{cases}$$

3.1. *One-sided exponential*

First we consider the case where the after-jump distribution is one-sided negative exponential. That is, we take

$$\phi(z) = \gamma e^{-\gamma z} \text{ for } z > 0. \tag{3.3}$$

For this case we find u, and in addition, we find, in explicit form, the distribution of $H(\infty) = \lim_{t\to\infty} H(t)$ where $H(\infty)$ is the amount of time the process spends above the threshold in $[0, \infty)$.

Using (3.3), we obtain

$$-\lambda\beta\xi(s)e^{-\gamma bs}/\alpha(\alpha+\beta)$$

for the right-hand side of (3.2). Introduce the one-sided Fourier transforms

$$\hat{u}^{\pm}(k) = \int_{-\infty}^{\infty} e^{iks}u^{\pm}(s)ds,$$

and apply this transform to (3.2), first noting that

$$\int_{-\infty}^{\infty} e^{iks}\frac{\partial u(s)}{\partial s}ds = -ik[\hat{u}^{+}(k) + \hat{u}^{-}(k)] + \frac{\beta}{\alpha(\alpha+\beta)},$$

to obtain

$$\begin{aligned}&[(\alpha+\beta)\mu - (\alpha+\beta+\lambda-\mu)ik + k^2]\hat{u}^{-}(k) + [\alpha\mu + (\alpha+\lambda-\mu)ik + k^2]\hat{u}^{+}(k)\\ &\qquad = \beta(\mu-\lambda-ik)/\alpha(\alpha+\beta)\end{aligned} \tag{3.4}$$

where $\mu = \gamma b$. Let

$$R_{1,2} = \tfrac{1}{2}[\mu - (\alpha+\beta+\lambda) \pm ([\mu-(\alpha+\beta+\lambda)]^2 + 4(\alpha+\beta)\mu)^{\frac{1}{2}}]$$

and T_1 and T_2 be equal to R_1 and R_2 respectively when $\beta = 0$. Note that $-iR_1$ and $-iR_2$ are roots of the coefficient of $\hat{u}^{-}$, and $-iT_1$ and $-iT_2$ are the roots of the coefficient of $\hat{u}^{+}$.

Let

$$K^{-}(k) = \frac{(k+iR_2)}{(k+iT_2)}, \qquad K^{+}(k) = \frac{(k+iT_1)}{(k+iR_1)}.$$

Dividing both sides of (3.4) by $(k+iR_1)(k+iT_2)$, and performing a partial fraction decomposition of the right-hand side of the resulting equation, we obtain

$$K^{-}(k)\hat{u}^{-}(k) - \frac{B}{k+iT_2} = -K^{+}(k)\hat{u}^{+}(k) + \frac{A}{k+iR_1}, \tag{3.5}$$

where

$$A = \frac{i\beta(\mu-\lambda-R_1)}{\alpha(\alpha+\beta)(T_2-R_1)}, \qquad B = \frac{i\beta(\mu-\lambda-T_2)}{\alpha(\alpha+\beta)(T_2-R_1)}.$$

In (3.5) the left-hand side is analytic in a lower half plane and the right-hand side is analytic in an overlapping upper half plane. The usual Wiener-Hopf argument (with an application of Liouville's theorem) shows that both sides of (3.5) are equal to zero. Solving for $\hat{u}^{-}$ and $\hat{u}^{+}$ and inverting, we obtain

$$u^+(s) = -iAe^{-T_1 s} \quad \text{for } s \geqq 0,$$

$$u^-(s) = iBe^{-R_2 s} \quad \text{for } s < 0.$$

Thus,

$$u(s) = \begin{cases} \dfrac{1}{\alpha}\left[1 + \dfrac{\beta}{\alpha+\beta}\dfrac{(\mu-\lambda-R_1)}{(R_1-T_2)}e^{-T_1 s}\right] & \text{for } s \geqq 0 \\ \dfrac{1}{\alpha+\beta}\left[1 + \dfrac{\beta}{\alpha}\dfrac{(\mu-\lambda-T_2)}{(R_1-T_2)}e^{-R_2 s}\right] & \text{for } s < 0. \end{cases}$$

Reintroducing the variable x in the argument of u, we note that $u(x, s) = u(0, s - x/b)$. Thus,

$$(3.6) \qquad u(x,0) = \begin{cases} \dfrac{1}{\alpha+\beta}\left[1 + \dfrac{\beta}{\alpha}\dfrac{(\mu-\lambda-T_2)}{(R_1-T_2)}e^{R_2 x/b}\right] & \text{for } x > 0 \\ \dfrac{1}{\alpha}\left[1 + \dfrac{\beta}{\alpha+\beta}\dfrac{(\mu-\lambda-R_1)}{(R_1-T_2)}e^{T_1 x/b}\right] & \text{for } x \leqq 0. \end{cases}$$

Note that

$$v(x) = \lim_{\beta\to\infty} u(x,0) = \tfrac{1}{2}\left[1 - \frac{\lambda}{\mu - T_2}e^{T_1 x/b}\right] \qquad \text{for } x \leqq 0.$$

One can invert $v(x)$ to obtain explicitly $\Pr\{X_s \leqq bs \text{ for } 0 \leqq s \leqq t \mid X_0 = x\}$. This probability has been found previously by Arfwedson (1950).

Although we are not able to invert u to obtain the distribution of $H(t)$, we are able to obtain the distribution of $H(\infty)$ as follows. Observe that

$$(3.7) \qquad \lim_{\alpha\to 0} \alpha u(0,0) = 1 + \frac{\mu-\lambda+\beta-([\beta-(\mu-\lambda)]^2+4\beta\mu)^{\frac{1}{2}}}{-\beta+([\beta-(\mu-\lambda)]^2+4\beta\mu)^{\frac{1}{2}}+|\mu-\lambda|}$$

is the transform of the generalized function g such that

$$\int_0^\tau g(t)dt = \Pr\{H(\infty) \leqq \tau \mid X_0 = 0\}.$$

If $\lambda \geqq \mu$, then the right-hand side of (3.7) equals 0, which corresponds to the fact that $H(\infty) = \infty$ with probability 1. Suppose $\lambda < \mu$, then

$$\lim_{\alpha\to 0} \alpha u(0,0) = -\frac{(\mu-\lambda)}{2\beta\mu}[\beta+\mu+\lambda-((\beta+\mu+\lambda)^2-4\lambda\mu)^{\frac{1}{2}}-2(\lambda+\beta)].$$

From (29.3.58) of Abramowitz and Stegun (1964), we find that $s - (s^2 - a^2)^{\frac{1}{2}}$ is the transform of $(a/\tau)I_1(a\tau)$. Thus

$$\beta+\mu+\lambda-((\beta+\mu+\lambda)^2-4\lambda\mu)^{\frac{1}{2}}$$

is the transform of

$$\frac{1}{\tau}2\sqrt{\lambda\mu}I_1(2\sqrt{\lambda\mu\tau})e^{-(\mu+\lambda)\tau}$$

and

$$g(\tau)=\frac{\mu-\lambda}{\mu}\{\delta(\tau)+\lambda-\sqrt{\lambda u}\int_0^\tau y^{-1}e^{-(\mu+\lambda)y}I_1(2y\sqrt{\lambda\mu})dy\}. \tag{3.8}$$

3.2. *Bilateral exponential*

We now consider the case in which ϕ is the bilateral exponential density

$$\phi(z)=\tfrac{1}{2}\gamma e^{-\gamma|z|},\qquad -\infty<z<\infty.$$

A straightforward computation shows that with this choice of ϕ the right-hand side of (3.2) becomes

$$-\frac{\lambda\beta}{2\alpha(\alpha+\beta)}\operatorname{sgn}(s)e^{-\gamma b|s|}.$$

Applying one-sided Fourier transforms to (3.2), we obtain

$$(\alpha+\beta+\lambda)\hat{u}^-(k)+(\alpha+\lambda)\hat{u}^+(k)-\lambda b[\hat{u}^+(k)+\hat{u}^-(k)]\left(\frac{\gamma^2 b}{k^2+\gamma^2b^2}\right)$$

$$+ik[\hat{u}^+(k)+\hat{u}^-(k)]-\frac{\beta}{\alpha(\alpha+\beta)}=-\frac{\lambda\beta}{\alpha(\alpha+\beta)}\left(\frac{ik}{k^2+\gamma^2b^2}\right).$$

For convenience of notation we take $\mu=\gamma b$ and multiply through by $-i(k^2+\mu^2)$ to obtain

$$[k^3-i(\alpha+\beta+\lambda)k^2+\mu^2k-i(\alpha+\beta)\mu^2]\hat{u}^-(k)$$

$$+[k^3-i(\alpha+\lambda)k^2+\mu^2k-i\alpha\mu^2]\hat{u}^+(k)=-\frac{i\beta}{\alpha(\alpha+\beta)}[k^2-i\lambda k+\mu^2]. \tag{3.9}$$

To assist in a Wiener-Hopf factorization we examine the roots of the various coefficient factors in this last equation. First, we have

$$k^2-i\lambda k+\mu^2=(k-ik_1)(k+ik_2), \tag{3.10}$$

where

$$k_1=\tfrac{1}{2}(\lambda+(\lambda^2+4\mu^2)^{\frac{1}{2}}),\ k_2=-\tfrac{1}{2}(\lambda-(\lambda^2+4\mu^2)^{\frac{1}{2}})$$

are both positive. The roots of the cubic

$$c(k)=k^3-i(\alpha+\beta+\lambda)k^2+\mu^2k-i(\alpha+\beta)\mu^2$$

are all pure imaginary, two being positive pure imaginary. To see this set $k=i\kappa$ and note that

$$ic(\pm i\mu)=-\lambda\mu^2,\qquad ic(0)>0,$$

and

$$ic(i[\alpha+\beta])<0,\ \lim_{k\to+\infty}ic(i\kappa)>0.$$

We write

$$c(k) = k^3 - i(\alpha+\beta+\lambda)k^2 + \mu^2 k - i(\alpha+\beta)\mu^2 \tag{3.11}$$
$$= (k - i\sigma_1)(k - i\sigma_2)(k + i\sigma_3),$$

where $\sigma_1, \sigma_2, \sigma_3$, are all positive real, with $\sigma_1, \sigma_3 < \mu$ and $\sigma_1 < \alpha + \beta < \sigma_2$. Further, we write

$$k^3 - i(\alpha+\lambda)k^2 + \mu^2 k - i\alpha\mu^2 = (k - i\rho_1)(k - i\rho_2)(k + i\rho_3), \tag{3.12}$$

where $\rho_i = \lim_{\beta\to 0}\sigma_i$, $i = 1,2,3$. For future reference we note that

$$\lim_{\alpha\to 0}\rho_1 = 0, \ \lim_{\alpha\to 0}\rho_2 = k_1, \text{ and } \lim_{\alpha\to 0}\rho_3 = k_2.$$

Using these various factorizations, we can write (3.9) in the form

$$\frac{(k-i\sigma_1)(k-i\sigma_2)}{(k-i\rho_1)(k-i\rho_2)}\hat{u}^-(k) + \frac{(k+i\rho_3)}{(k+i\sigma_3)}\hat{u}^+(k) = -\frac{i\beta}{\alpha(\alpha+\beta)}\frac{(k-ik_1)(k+ik_2)}{(k-i\rho_1)(k-i\rho_2)(k+i\sigma_3)}.$$

To split the right-hand side we use partial fractions and obtain the equation

$$\frac{(k-i\sigma_1)(k-i\sigma_2)}{(k-i\rho_1)(k-i\rho_2)}\hat{u}^-(k) + \frac{iA}{k-i\rho_1} + \frac{iB}{k-i\rho_2} = -\frac{(k+i\rho_3)}{(k+i\sigma_3)}\hat{u}^+(k) - \frac{iC}{k+i\sigma_3}, \tag{3.13}$$

where

$$A = \frac{\beta}{\alpha(\alpha+\beta)}\frac{(\rho_1-k_1)(\rho_1+k_2)}{(\rho_1-\rho_2)(\rho_1+\sigma_3)},$$

$$B = \frac{\beta}{\alpha(\alpha+\beta)}\frac{(\rho_2-k_1)(\rho_2+k_2)}{(\rho_2-\rho_1)(\rho_2+\sigma_3)},$$

$$C = \frac{\beta}{\alpha(\alpha+\beta)}\frac{(\sigma_3+k_1)(\sigma_3-k_2)}{(\sigma_3+\rho_1)(\sigma_3+\rho_2)}.$$

The usual Wiener-Hopf argument shows that both sides of (3.13) are equal to zero. Hence

$$\hat{u}^+(k) = \frac{-iC}{k+i\rho_3} \text{ and } \hat{u}^-(k) = \frac{-iD}{k-i\sigma_1} + \frac{-iE}{k-i\sigma_2},$$

where a partial fraction decomposition is used to show that

$$D = ((A+B)\sigma_1 - (A\rho_2 + B\rho_1))/(\sigma_1-\sigma_2),$$

$$E = -((A+B)\sigma_2 - (A\rho_2 + B\rho_1))/(\sigma_1-\sigma_2).$$

Inverting these Fourier transforms we have

$$u^+(s) = -Ce^{-\rho_3 s} \text{ and } u^-(s) = De^{\sigma_1 s} + Ee^{\sigma_2 s},$$

or

$$u(s) = \begin{cases} -Ce^{-\rho_3 s} + (1/\alpha), & s \geqq 0 \\ De^{\sigma_1 s} + Ee^{\sigma_2 s} + (1/(\alpha+\beta)), & s < 0. \end{cases} \tag{3.14}$$

We now reintroduce x into the problem through the relation $u(x,s) = u(0,s-x/b)$. The transform of the probability that the process fails to cross the threshold in $[0,t]$ is computed as the limit

$$v(x) = \lim_{\beta\to\infty} u(x,0),$$

or

$$v(x) = \begin{cases} 0, & x > 0 \\ (1/\alpha) - \bar{C}e^{\rho_3 x/b}, & x \leqq 0. \end{cases}$$

Here

$$\bar{C} = \lim_{\beta\to\infty} C = \frac{1}{\alpha}\,\frac{(\mu+k_1)(\mu-k_2)}{(\mu+\rho_1)(\mu+\rho_2)},$$

where use has been made of the relations

$$\lim_{\beta\to\infty}\sigma_2 = \infty, \text{ and } \lim_{\beta\to\infty}\sigma_3 = \lim_{\beta\to\infty}\sigma_1 = \mu.$$

We now focus attention on the probability p of eventually crossing the threshold. This probability is given by

$$p = 1 - \lim_{\alpha\to 0}\alpha v(x).$$

Recalling the limits of the ρ_i as $\alpha \to 0$ we have at once

(3.15) $$p = \frac{(\mu+k_1)(\mu-k_2)}{\mu(\mu+k_1)}e^{k_2 x/b} = \left(1 - \frac{k_2}{\mu}\right)e^{k_2 x/b},$$

where

$$k_2 = -\tfrac{1}{2}(\lambda - (\lambda^2 + 4\mu^2)^{\frac{1}{2}}) > 0,$$

a result given as the solution to Problem 7 of Chapter 7 of Takács (1967).

Finally, as in Stone, Belkin and Snyder (1970), we can consider

$$v(0) = \lim_{\beta\to\infty} u(0,0),$$

the transform of $\Pr\{X_s \leqq bs \text{ for } 0 \leqq s \leqq t \mid X_0 = 0\}$. Here we have

$$v(0) = \frac{1}{\alpha}\left[1 - \frac{(\mu+k_1)(\mu-k_2)}{(\mu+\rho_1)(\mu+\rho_2)}\right].$$

Setting $k = -i\mu$ in both sides of (3.12) we obtain

$$(\mu+\rho_1)(\mu+\rho_2) = \lambda\mu^2/(\mu-\rho_3).$$

While the same substitution in (3.10) gives $(\mu+k_1)(\mu-k_2) = \lambda\mu$, so that

(3.16) $$v(0) = \rho_3/\alpha\mu,$$

where ρ_3 is the positive real root of the cubic

$$\rho^3 + (\alpha + \lambda)\rho^2 - \mu^2\rho - \alpha\mu^2 = 0.$$

4. Example 3: first passage distribution for an integrated Poisson sampling process

Suppose $\{X_t\}$ is a Poisson sampling process, i.e., $X_t = U_{N(t)}$, where $U_0, U_1, \cdots$, is a sequence of independent, identically distributed random variables with common distribution K and $\{N_t\}$ is a Poisson process of sampling times with parameter $\lambda > 0$. The distribution of time above a constant threshold for the process $\{X_t\}$ is computed in Stone, Belkin and Snyder (1970), and the first passage problem for a general threshold is considered in Belkin (1971). Here we will be concerned with the integrated Poisson sampling process $Y_t = Y_0 + \int_0^t X_s ds$, where Y_0 is a constant. One might, for example, interpret X_t as the net rate of accumulation at time t of some quantity (such as an industrial commodity, rainfall, etc.) under the assumption that the rates of production and consumption are each Poisson sampling processes. Then Y_t is simply the amount of the quantity available at time t and it is of interest to compute the distribution of the time until Y_t first exceeds (or, in the symmetric problem, falls below) a critical level.

It was observed earlier that while $\{Y_t\}$ is not itself a Markov process, the two-dimensional process $\{Z_t\} = \{X_t, Y_t\}$ is Markovian with stationary transition measure. Furthermore $\{Z_t\}$ has a stochastically continuous transition density and Theorem 1.2 applies. Without loss of generality we take $Y_0 = 0$.

If $f: R^2 \to R^1$ is in the domain of the weak infinitesimal operator $\mathscr{A}$ of Z_t, then a simple calculation shows

$$\mathscr{A}f(x,y) = x\,\frac{\partial f}{\partial y} - \lambda f(x,y) + \lambda \int_{-\infty}^{\infty} f(\xi, y)dK(\xi).$$

Suppose that

$$(\alpha - \hat{\mathscr{A}})w = I,$$

where $\Gamma = R^1 \times (l, \infty)$, $\hat{\mathscr{A}}$ is the weak infinitesimal operator of the process $\hat{Z}_t$ with extinction at the hitting time of Γ, and

$$I(x,y) = \begin{cases} 1, & y < l \text{ or } y = l \text{ and } x \leqq 0 \\ 0, & y > l \text{ or } y = l \text{ and } x > 0. \end{cases}$$

(The conditions at the threshold $y = l$ are chosen to insure that $\lim_{t \downarrow 0} \hat{T}_t I = I$.)

Using the properties that

(i) $E_{x,y}[\mu\{t: Z_t \in \partial\Gamma\}] = 0$, except for $(x,y) = (0,l)$,

(ii) $I(0,l) = 1$,

it can be directly verified that

$$\hat{R}_\alpha I(x,y) = v(x,y) = \int_0^{\infty} e^{-\alpha t} P_{x,y}[\tau_\Gamma > t]dt.$$

Furthermore it follows from

$$P[N(t) \geqq 1] = O(t) \text{ as } t \to 0,$$

that for $f: R^2 \to R^1$ in the domain of $\hat{\mathscr{A}}$ (and thus vanishing on Γ),

$$|\hat{T}_t f(x, y) - T_t f(x, y)| = o(t) \text{ as } t \to 0$$

and hence that $\hat{\mathscr{A}} f = \mathscr{A} f$ on the common domain of the operators. Consequently, by Theorem 1.2, v is the unique bounded Borel measurable function which vanishes on Γ and satisfies

$$(4.1) \qquad (\alpha + \lambda)v - x\frac{\partial v}{\partial y} - \lambda \int_{-\infty}^{\infty} v(\xi, y) dK(\xi) = I(x, y).$$

The solution of the integro-differential equation in (4.1) appears quite difficult in general and we therefore specialize to the case in which K is a two point distribution with masses p $(0 < p < 1)$ at x_1 and $q = 1 - p$ at x_2. We assume $x_1 > x_2$ and restrict attention to the non-trivial case $x_1 > 0$. The expression in (4.1) leads to the system of first order ordinary differential equations (we assume from now on, unless otherwise noted, that $y < l$)

$$(4.2) \qquad \begin{aligned} (\alpha + \lambda)v_1 - x_1 v_1 - \lambda(pv_1 + qv_2) &= 1, \\ (\alpha + \lambda)v_2 - x_2 v_2' - \lambda(pv_1 + qv_2) &= 1, \end{aligned}$$

where $v_i(y) = v(x_i, y)$, $i = 1, 2$. Direct substitution in (4.2) then gives the second order differential equation

$$(4.3) \qquad x_1 x_2 v_1'' - [\alpha(x_1 + x_2) + \lambda(px_1 + qx_2)]v_1' + \alpha(\alpha + \lambda)v_1 = \alpha + \lambda,$$

with the boundary conditions $v_1(l) = 0$ and v_1 bounded at $-\infty$.

The general solution to (4.3) has the form

$$(4.4) \qquad v_1(y) = c_1 e^{r_+ y} + c_2 e^{r_- y} + (1/\alpha),$$

where

$$r_\pm = [\alpha(x_1 + x_2) + \lambda(px_1 + qx_2) \\ \pm ([\alpha(x_1 + x_2) + \lambda(px_1 + qx_2)]^2 - 4x_1 x_2 \alpha(\alpha + \lambda))^{\frac{1}{2}}]/2x_1 x_2 .$$

The problem now divides into two separate cases depending on the sign of x_2.

Case 1: $x_2 < 0$. In this case r_- is positive and r_+ negative. The boundary conditions on v_1 then require that

$$c_1 = 0, \qquad c_2 = -e^{-r_- l}/\alpha,$$

and hence that

$$(4.5) \qquad v_1(y) = [1 - e^{-r_-(l-y)}]/\alpha.$$

Now the radical in the expression for r_- may be written in the form

$$([\alpha(x_1 - x_2) + \lambda(px_1 - qx_2)]^2 + 4\lambda^2 pqx_1x_2)^{\frac{1}{2}} = (x_1 - x_2)(\alpha + a)^{\frac{1}{2}}(\alpha + b)^{\frac{1}{2}},$$

where

$$a = \lambda \frac{px_1 - qx_2}{x_1 - x_2} + \frac{2\lambda\sqrt{-pqx_1x_2}}{x_1 - x_2},$$

$$b = \lambda \frac{px_1 - qx_2}{x_1 - x_2} - \frac{2\lambda\sqrt{-pqx_1x_2}}{x_1 - x_2}.$$

Finding the inverse transform of v_1 requires the inversion of

$$\exp\left\{ -\frac{x_1 - x_2}{2x_1|x_2|}(l - y)(\alpha + a)^{\frac{1}{2}}(\alpha + b)^{\frac{1}{2}}\right\}.$$

Using (863.1) of Campbell and Foster (1948) the inverse transform of $\exp\{ - c(\alpha + a)^{\frac{1}{2}}(\alpha + b)^{\frac{1}{2}}\}$ is given as

$$H_1(t) = \tfrac{1}{2}(a - b)\frac{c}{(t^2 - c^2)^{\frac{1}{2}}} e^{-\frac{1}{2}(a+b)t} I_1(\tfrac{1}{2}(a - b)(t^2 - c^2)^{\frac{1}{2}}) + e^{-\frac{1}{2}c(a+b)}\delta(t - c), \; a \geqq b, \; t \geqq c, \tag{4.6}$$

where I_1 is the modified Bessel function of the first order and δ is the usual Dirac δ-function. Since u_i is a function of $l - y$ it suffices to restrict attention to the case $y = 0$, $l \geqq 0$. Define $G_i(t, l) = 1 - P_{x_i,0}[Y_s \leqq l, 0 \leqq s \leqq t]$, $i = 1,2$. Then it follows from (4.5) and (4.6) that G_1 has a density g_1 given by

$$g_1(t, l) = \exp\left\{ -\lambda\frac{px_1 + qx_2}{2x_1x_2} l\right\} H_1\left(t + l\,\frac{x_1 + x_2}{2x_1x_2}\right) \quad \text{for } t \geqq l/x_1, \tag{4.7}$$

where the parameter c in (4.6) is taken as

$$c = \frac{x_1 - x_2}{2x_1|x_2|} l.$$

A straightforward evaluation of (4.7) then shows that

$$g_1(t, l) = \frac{\lambda l\sqrt{pq}}{(x_1t - l)^{\frac{1}{2}}(l - x_2t)^{\frac{1}{2}}} \exp\left\{ -\frac{\lambda}{x_1 - x_2}[p(x_1t - l) + q(l - x_2t)]\right\} \times I_1\left(\frac{2\lambda\sqrt{pq}}{x_1 - x_2}(x_1t - l)^{\frac{1}{2}}(l - x_2t)^{\frac{1}{2}}\right) + e^{-\lambda ql/x_1}\,\delta(t - l/x_1) \quad \text{for } t \geqq l/x_1. \tag{4.8}$$

One possible way to obtain g_2 is to observe the following relationship:

$$g_2(t, l) = \lambda p \int_0^{(x_1t - l)/(x_1 + x_2)} e^{-\lambda ps}\, f_1(t - s, l + sx_2)ds, \tag{4.9}$$

which is based on conditioning on the first time s that $X_s = x_1$. Substituting the expression for g_1 in (4.8) into (4.9) and simplifying gives

$$
\begin{aligned}
g_2(t, l) &= \frac{\lambda p}{l - x_2 t} \exp \left\{ -\frac{\lambda}{x_1 - x_2} [p(x_1 t - l) + q(l - x_2 t)] \right\} \\
&\times \left\{ l I_0 \left(\frac{2\lambda\sqrt{pq}}{x_1 - x_2} (x_1 t - l)^{\frac{1}{2}} (l - x_2 t)^{\frac{1}{2}} \right) \right. \\
&\left. - \frac{x_2}{\lambda\sqrt{pq}} \left(\frac{x_1 t - l}{l - x_2 t} \right)^{\frac{1}{2}} I_1 \left(\frac{2\lambda\sqrt{pq}}{x_1 - x_2} (x_1 t - l)^{\frac{1}{2}} (l - x_2 t)^{\frac{1}{2}} \right) \right\}.
\end{aligned} \tag{4.10}
$$

Case 2: $x_2 > 0$. Now both r_+ and r_- are positive and the solution to (4.3) takes the form

$$v_1(y) = c_1 e^{r_+ y} \quad c_2 e^{r_- y} + (1/\alpha)$$

with boundary conditions $v_1(l) = v_2(l) = 0$. It follows that

$$c_1 = \frac{x_1 r_- - \alpha}{\alpha x_1 (r_+ - r_-)} e^{-r_+ l}$$

and

$$c_2 = \frac{\alpha - x_1 r_+}{\alpha x_1 (r_+ - r_-)} e^{-r_- l}.$$

Thus

$$v_1(y) = \frac{1}{\alpha} \left[1 - \left\{ \frac{\alpha - x_1 r_-}{x_1 (r_+ - r_-)} e^{-r_+(l-y)} - \frac{\alpha - x_1 r_+}{x_1 (r_+ - r_-)} e^{-r_-(l-y)} \right\} \right]. \tag{4.11}$$

Again it suffices to consider only the case $y = 0$, $l \geqq 0$. Now the crucial expression to invert is

$$
\begin{aligned}
(\alpha + a)^{-\frac{1}{2}} (\alpha + b)^{-\frac{1}{2}} \left(\alpha + \lambda \frac{p x_1 + q x_2}{x_1 - x_2} \right) \sinh \left(\frac{x_1 - x_2}{2 x_1 x_2} (\alpha + a)^{\frac{1}{2}} (\alpha + b)^{\frac{1}{2}} l \right) \\
+ \cosh \left(\frac{x_1 - x_2}{2 x_1 x_2} (\alpha + a)^{\frac{1}{2}} (\alpha + b)^{\frac{1}{2}} l \right),
\end{aligned}
$$

where

$$a = \lambda \left(\frac{p x_1 - q x_2}{x_1 - x_2} + \frac{2\sqrt{pq x_1 x_2}}{x_1 - x_2} i \right)$$

and

$$b = \lambda \left(\frac{p x_1 - q x_2}{x_1 - x_2} - \frac{2\sqrt{pq x_1 x_2}}{x_1 - x_2} i \right).$$

The appropriate (Fourier) transform pairs (again see (871.3) and (872.2) of Campbell and Foster (1948)) are

$$\cosh(c(\alpha + a)^{\frac{1}{2}}(\alpha + b)^{\frac{1}{2}}) \to$$

$$-\frac{c(a-b)}{4(c^2-t^2)^{\frac{1}{2}}} e^{-\frac{1}{2}(a+b)t} J_1(\tfrac{1}{2}(a-b)(c^2-t^2)^{\frac{1}{2}}) + \tfrac{1}{2}e^{\frac{1}{2}(a+b)c}\,\delta(t+c)$$
$$+ \tfrac{1}{2}e^{-\frac{1}{2}(a+b)c}\delta(t-c) \text{ for } c > 0, |t| \leqq c,$$

and

$$\frac{\sinh(c(\alpha+a)^{\frac{1}{2}}(\alpha+b)^{\frac{1}{2}})}{(\alpha+a)^{\frac{1}{2}}(\alpha+b)^{\frac{1}{2}}} \to \tfrac{1}{2}e^{-\frac{1}{2}(a+b)t} J_0(\tfrac{1}{2}(a-b)(c^2-t^2)^{\frac{1}{2}}) \text{ for } c > 0, |t| \leqq c,$$

where J_0 and J_1 are ordinary Bessel functions.

Define

$$H_1(t) = \frac{qx_2}{x_1 - x_2} e^{-\frac{1}{2}(a+b)t} J_0(\tfrac{1}{2}(a-b)(c^2-t^2)^{\frac{1}{2}})$$
$$+ \frac{(a-b)(t-c)}{4(c^2-t^2)^{\frac{1}{2}}} e^{-\frac{1}{2}(a+b)t} J_1(\tfrac{1}{2}(a-b)(c^2-t^2)^{\frac{1}{2}})$$
$$+ e^{\frac{1}{2}(a+b)c}\,\delta(t+c) \text{ for } |t| \leqq c,$$

where

$$c = \frac{x_1 - x_2}{2x_1x_2} l.$$

Equation (4.7) again applies except that the region of validity is now $l/x_1 \leqq t \leqq l/x_2$. Simplification then leads to the expression

$$\begin{aligned} g_1(t,l) = \lambda q \exp&\left\{-\frac{\lambda}{x_1-x_2}[p(x_1t-l)+q(l-x_2t)]\right\} \\ &\times \left\{\frac{x_2}{x_1-x_2} I_0\left(\frac{2\lambda\sqrt{pq}}{x_1-x_2}(x_1t-l)^{\frac{1}{2}}(l-x_2t)^{\frac{1}{2}}\right)\right. \\ &\left. + \frac{x_1}{x_1-x_2}\left(\frac{p(l-x_2t)}{q(x_1t-l)}\right)^{\frac{1}{2}} I_1\left(\frac{2\lambda\sqrt{pq}}{x_1-x_2}(x_1t-l)^{\frac{1}{2}}(l-x_2t)^{\frac{1}{2}}\right)\right\} \\ &+ e^{-\lambda ql/x_1}\,\delta(t-l/x_1), \qquad l/x_1 \leqq t \leqq l/x_2. \end{aligned} \tag{4.12}$$

One can obtain g_2 analogously or, using the natural symmetry of the problem, by simply interchanging p with q and x_1 with x_2 (except in the difference $x_1 - x_2$) in (4.12).

The first passage densities g_1 and g_2 when $x_2 = 0$ obtained by solving (4.3) agree with the appropriate limits in (4.8), (4.10), and (4.12). Thus expressions for g_1 and g_2 have been obtained for all $x_2 < x_1$.

The transforms in (4.5) and (4.11) can also be used directly to compute first moments and investigate the asymptotic tail behavior of the distributions G_1 and G_2.

Let $\mu_i = E_{x_i,0}[\tau_\Gamma]$, the expected time of first passage to the level l for the integrated process. Then μ_i can be obtained as the limit

$$\mu_i = \lim_{\alpha \downarrow 0} u_i(0).$$

A simple Taylor series development of v_i in powers of α shows that μ_i is finite only when $px_1 + qx_2$, the stationary mean of the process X_t, is positive in which case

$$(4.13)\quad \begin{aligned} \mu_1 &= l/(px_1 + qx_2) \text{ when } x_2 \leqq 0, \\ \mu_1 &= \frac{q(x_1 - x_2)x_2}{\lambda(px_1 + qx_2)^2}\left[\exp\left\{-\lambda\frac{px_1 + qx_2}{x_1 x_2} l\right\} - 1\right] + \frac{l}{px_1 + qx_2} \text{ when } x_2 > 0. \end{aligned}$$

(Note the interesting property that μ_1 is independent of λ.)

Similarly,

$$(4.14)\quad \begin{aligned} \mu_2 &= \frac{l}{px_1 + qx_2}\left(l + \frac{x_1 - x_2}{\lambda}\right) \text{ when } x_2 \leqq 0, \\ \mu_2 &= \frac{p(x_1 - x_2)x_1}{\lambda(px_1 + qx_2)^2}\left[\exp\left\{-\lambda\frac{px_1 + qx_2}{x_1 x_2} l\right\} - 1\right] + \frac{l}{px_1 + qx_2} \text{ when } x_2 > 0. \end{aligned}$$

When $px_1 + qx_2 = 0$ an application of Karamata's Tauberian theorem (see Widder (1941)) shows that

$$(4.15)\quad \begin{aligned} 1 - G_1(t, l) &\sim \frac{l}{x_1 |x_2|}\left(\frac{\lambda(x_1 - x_2)(px_1 - qx_2)}{2\pi t}\right)^{\frac{1}{2}} \text{ at } t \to \infty, \\ 1 - G_2(t, l) &\sim \left[\frac{1}{\lambda q |x_2|} + \frac{l}{x_1 |x_2|}\right]\left(\frac{\lambda(x_1 - x_2)(px_1 - qx_2)}{2\pi t}\right)^{\frac{1}{2}} \text{ at } t \to \infty. \end{aligned}$$

Finally, when $px_1 + qx_2 < 0$, the first passage distributions are defective with

$$(4.16)\quad \begin{aligned} \lim_{t \to \infty} 1 - G_1(t, l) &= 1 - \exp\left\{-\frac{\lambda(qx_2 + px_1)l}{x_1 x_2}\right\}, \\ \lim_{t \to \infty} 1 - G_2(t, l) &= 1 - \frac{px_1}{q|x_2|}\exp\left\{-\frac{\lambda(qx_2 + px_1)l}{x_1 x_2}\right\}. \end{aligned}$$

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