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Distribution of Time above a Threshold for Markov Processes*

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1. INTRODUCTION

In this paper, we are primarily concerned with the problem of finding the distribution of the time a Markov process, $\{X_t : t \geqslant 0\}$, with stationary transition probabilities spends above a fixed level in a given time interval. We let $H_l(t)$ be the amount of time the process spends at or above the level l in the interval $[0, t]$. Then we wish to find $F_x(t, \tau) = P[H_l(t) \leqslant \tau \mid X_0 = x]$. We attack this problem by finding an equation whose unique solution is the double transform of F_x.

The time above a threshold problem arises quite naturally in the context of communication when signal strength is assumed to be a Markov process. Assuming that communication is possible only when the signal strength is above a threshold l, $F_x(t, \tau)$ gives the distribution of the amount of time in $[0, t]$ during which communication is possible. In general, knowledge of F_x will be of interest whenever it is of importance to measure the cumulative time a stochastic process (representing possibly radiation, noise, or pressure) spends above a critical level.

The time above a threshold problem has been considered in the context of the fluctuations of a random walk. Let N_n be the number of positive partial sums among the first n partial sums in a random walk. It has been shown by Anderson [1] and Spitzer [2] that if the increments of the random walk have mean zero and finite variance, then N_n/n has a limiting arcsine distribution.

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When $\{X_t\}$ is a centered separable process with stationary independent increments such that $X_0 = 0$, Wendel [3] found the Laplace transform of the joint characteristic function of $H_l(t)$ and X_t. He also showed that if $P[X_t > 0]$ is independent of t, then $H_0(t)/t$ has an arcsine distribution.

Most work on the time above a threshold problem has concentrated on the special case where $l = x = E_x[X_t]$. However, for the standard Wiener process, the distribution of the time above an arbitrary threshold is an easy consequence of the case $l = x = 0$. (See, e.g., Ito and Mc Kean [4] and Kac [5] for this case.) The distribution of time above an arbitrary threshold has in effect been found for a process which corresponds to independent draws from a fixed distribution at exponentially distributed random times. The latter has been solved in the context of reliability theory as the distribution of machine down time (see Barlow and Proschan [6]) using a result of Takacs [7].

The moments of the amount of time a stochastic process spends above a curve have also been considered (e.g., Leadbetter and Cryer [8]).

In this paper we consider homogeneous, additive functionals of the form

$$H(t) = \int_0^t h(X_s)\, ds,$$

where h is a bounded, nonnegative, Borel function. In the case where $h(x) = 1$ for $x \geqslant l$ and 0 otherwise, $H(t)$ becomes $H_l(t)$. We treat the double transform

$$u(x) = \int_0^\infty e^{-\alpha t} E_x[e^{-\beta H(t)} f(X_t)]\, dt,$$

where f is a bounded Borel function and E_x denotes expectation conditioned on $X_0 = x$. In Theorem 2.2 it is shown that for a measurable Markov process with stationary transition probabilities, u is a solution of

$$(\alpha - \mathscr{A} + \beta h)\, u = f, \tag{1.1}$$

where $\mathscr{A}$ is the weak infinitesimal generator of the Markov process. Conditions are given which guarantee that u is the unique bounded Borel solution of (1.1).

We remark that the conclusion of Theorem 2.2 has been obtained by Dynkin (Theorem 9.7 of [9]) in the context of transforming resolvent and weak infinitesimal operators for a more restricted class of Markov processes. In particular, one additional restriction imposed by Dynkin is that the process be strong Markov.

A result of a type similar to that of Theorem 2.2 has been found by Darling and Siegert in [10]. They show that for a stationary measurable Markov process $\{X_t\}$,

$$R(x, \Gamma) = \int_0^\infty e^{-st} E[e^{i\xi U(t)} \mid X_0 = x \text{ and } X_t \in \Gamma] \, dt$$

is the unique solution of a pair of integral equations, where Γ is a Borel set of real numbers and $U(t)$ is defined in the same way as $H(t)$ without the restriction that h be nonnegative. Finding the distribution of $U(t)$ has been studied by Beekman [11] in the case of an Ornstein-Uhlenbeck process.

The method of proving Theorem 2.2 used here is a generalization of the method used in [4] to find u for a Wiener process. However, Ito and Mc Kean made substantial use of the form of $\mathscr{A}$ for a Wiener process to prove the uniqueness of the solution of (1.1). Our main departure from the proof in [4] is to prove uniqueness of the solution directly from the general properties of the weak infinitesimal operator of a Markov process.

For a specific Markov process the nature of the functional equation to be solved for u will depend principally on the form of $\mathscr{A}$, and typically one may have to solve integro-differential or difference equations to find u. Nevertheless, once the form of $\mathscr{A}$ is known the problems of finding u and inverting it to obtain F_α are strictly analytic in nature.

In Sections 3 to 6, we apply the results of Section 2 to find the double transform of F_α for a process which may be thought of as sampling at random times, for randomized simple random walk, for the Ornstein-Uhlenbeck process, and for a special compound Poisson process. For sampling at random times the inversion is performed to find F_α. In the other examples a limiting procedure is used to find the transform of the first passage time distribution.

2. Main Theorem

Let $\{X_t : t \geqslant 0\}$ be a Markov process with stationary transition probabilities defined on a probability space $(\Omega, \mathfrak{A}, P)$ with state space $(S, \mathscr{S})$, where $\mathscr{S}$ is the σ algebra of Borel subsets of S. To the pair $(S, \mathscr{S})$, we associate the Banach space B of real-valued bounded Borel measurable functions f on S with norm

$$\|f\| = \sup_{x \in S} |f(x)| .$$

Convergence is assumed to be weak convergence, i.e., $\lim_{n\to\infty} f_n = f$ if

(a) $\lim_{n\to\infty} f_n(x) = f(x)$, for each $x \in S$,

(b) $\|f_n\|$ are bounded.

Let P defined on $[0, \infty) \times S \times \mathscr{S}$ be the transition function of $\{X_t\}$, and define $T_t : B \to B$ by

$$T_t f(x) = \int_S P(t, x, dy) f(y).$$

Define $\tilde{B}$ to be the set of all $f \in B$ such that

$$\lim_{t \to 0+} T_t f = f.$$

The weak infinitesimal operator $\mathscr{A}$ is defined by

$$\mathscr{A} f = \lim_{t \to 0+} \frac{T_t f - f}{t}$$

whenever the limit on the right side exists in the weak sense and is a member of $\tilde{B}$. For each $\alpha > 0$ define the resolvent operator $R_\alpha : B \to B$ as follows:

$$R_\alpha g(x) = \int_0^\infty e^{-\alpha t} T_t g(x)\, dt. \tag{2.1}$$

For future reference, we paraphrase Theorem 1.7 from [9].

THEOREM 2.1. *For any $\alpha > 0$, the operator $\alpha - \mathscr{A}$ is a one-to-one mapping of the domain of $\mathscr{A}$ onto $\tilde{B}$. The inverse operator $(\alpha - \mathscr{A})^{-1}$ is the restriction of R_α to $\tilde{B}$.*

In what follows we shall assume that $\{X_t\}$ is a measurable Markov process, i.e., the mapping X: $[0, \infty) \times \Omega \to S$ is measurable.

For $h \geqslant 0$ in B let

$$H(t, \omega) = \int_0^t h[X_s(\omega)]\, ds. \tag{2.2}$$

For each $s \geqslant 0$ and $\omega \in \Omega$, let ω_s^+ be such that $X_t(\omega_s^+) = X_{t+s}(\omega)$. For $0 \leqslant s \leqslant t$, it is clear that

$$H(t, \omega) = H(s, \omega) + H(t - s, \omega_s^+). \tag{2.3}$$

If f is in B, then for $\alpha > 0$ and $\beta > 0$ let

$$u(x) = E_x \left[\int_0^\infty e^{-\alpha t} e^{-\beta H(t)} f(X_t)\, dt \right]. \tag{2.4}$$

The following theorem gives a generalization of Kac's formula for Brownian functionals as given in [4] and is the principal result of the paper.

THEOREM 2.2. *Let $\{X_t\}$ be a measurable Markov process with stationary transition probabilities and weak infinitesimal operator $\mathscr{A}$. If f and hu are in $\tilde{B}$, with u as defined in* (2.4), *then u is the unique bounded Borel measurable solution of*

$$(\alpha - \mathscr{A} + \beta h)\, u = f, \tag{2.5}$$

for $\alpha > \beta \| h \|$.

Proof. We first show that u satisfies (2.5) by following the proof in [4]. We have

$$\begin{aligned} u(x) - R_\alpha f(x) &= E_x \left\{ \int_0^\infty e^{-\alpha t}[e^{-\beta H(t)} - 1]\, f(X_t)\, dt \right\} \\ &= - E_x \left[\int_0^\infty e^{-\alpha t} \int_0^t e^{-\beta H(t-s, \omega_s^+)} \beta H(ds)\, f(X_t)\, dt \right] \\ &= - E_x \left[\int_0^\infty \beta H(ds) \int_s^\infty e^{-\alpha t} e^{-\beta H(t-s, \omega_s^+)} f(X_t)\, dt \right], \end{aligned}$$

where the interchange of the order of integration is justified by Fubini's theorem. Consequently,

$$u(x) - R_\alpha f(x) = - E_x \left\{ \int_0^\infty e^{-\alpha s}\ \mathscr{A}(ds) \int_0^\infty e^{-\alpha t} e^{-\beta H(t, \omega_s^+)} f[X_t(\omega_s^+)]\, dt \right\}.$$

Taking conditional expectation with respect to X_s inside of E_x, we have

$$\begin{aligned} u(x) - R_\alpha f(x) &= - E_x \left[\int_0^\infty e^{-\alpha s} \beta h(X_s)\, u(X_s)\, ds \right] \\ &= - R_\alpha \beta h u(x). \end{aligned}$$

That is, $u = R_\alpha(f - \beta h u)$. By virtue of Theorem 2.1, u is in the domain of $\mathscr{A}$ and $(\alpha - \mathscr{A})\, u = f - \beta h u$, proving (2.5).

We finish the proof by showing that for $\alpha > \beta \| h \|$, u is the unique bounded Borel measurable solution of (2.5). Let $\hat{u}$ be a bounded Borel measurable solution of (2.5). Since $\hat{u}$ is in the domain of $\mathscr{A}$, it is in $\tilde{B}$. From (2.5) we have that $h\hat{u} = (f + \mathscr{A}\hat{u} - \alpha\hat{u})/\beta$ is in $\tilde{B}$. We may write (2.5) as

$$(\alpha - \mathscr{A})\, \hat{u} = f - \beta h \hat{u}.$$

Since $f - \beta h\hat{u}$ is in $\tilde{B}$, by Theorem 2.1,

$$\hat{u} = R_\alpha(\alpha - \mathscr{A})\, \hat{u} = R_\alpha f - \beta R_\alpha h \hat{u}.$$

Therefore,

$$(1 + \beta R_\alpha h)\, \hat{u} = R_\alpha f.$$

Since $\| R_\alpha \| < 1/\alpha$, we have that for $\alpha > \beta \| h \|$, $(1 + \beta R_\alpha h)$ is invertible and $u = \hat{u}$. This concludes the proof.

The following corollary is useful in finding u for many Markov processes.

COROLLARY 2.1. *Let $\{X_t\}$ be a Markov process with stationary transition probabilities and right-continuous sample paths such that*

$$\lim_{t \to 0+} P(t, x, \{x\}) = 1.$$

Let f and h be in B. Then u as defined in (2.4) *is the unique bounded Borel measurable solution of* (2.5) *for $\alpha > \beta \| h \|$.*

Proof. Since $\{X_t\}$ has right-continuous sample paths, it is a measurable process.

The assumption that $\lim_{t \to 0+} P(t, x, \{x\}) = 1$ guarantees that for any bounded Borel measurable function g,

$$\lim_{t \to 0+} T_t g(x) = g(x).$$

Thus, hu is in $\tilde{B}$, and by Theorem 2.2, u is the unique bounded Borel measurable solution of (2.5). This proves the corollary.

COROLLARY 2.2. *Let $\{X_t\}$ be a Markov process with stationary transition probabilities and right-continuous sample paths. If u is a bounded continuous function and h and f are in $\tilde{B}$, then for $\alpha > \beta \| h \|$, u is the unique bounded Borel measurable solution of* (2.5).

Proof. Since $\{X_t\}$ has right-continuous sample paths it is a measurable process, and, in order to apply Theorem 2.2, we need only show that hu is in $\tilde{B}$. Since h is in $\tilde{B}$, it is sufficient to show that

$$\lim_{t \to 0+} | T_t hu(x) - u(x) T_t h(x)| = 0.$$

However,

$$\begin{aligned} | T_t hu(x) - u(x) T_t h(x)| &= | E_x h(X_t) (u(X_t) - u(x))| \\ &\leqslant \| h \| E_x | u(X_t) - u(x)| \to 0 \end{aligned}$$

by the right continuity of $\{X_t\}$ and the continuity of u. This proves the corollary.

We are primarily interested in the special case of Theorem 2.2 when $f \equiv 1$, $h(x) = 0$ for $x < l$ and $h(x) = 1$ for $x > l$. (The appropriate value of $h(l)$ depends on the process in question.) Since $F_x(t, \tau) = P[H_l(t) \leqslant \tau \mid X_0 = x]$, Eq. (2.4) becomes

$$u(x) = \int_0^\infty \int_0^\infty e^{-\alpha t} e^{-\beta \tau} F_x(t, d\tau)\, dt, \tag{2.6}$$

with u satisfying

$$(\alpha - \mathscr{A} + \beta h)\, u = 1. \tag{2.7}$$

We note that knowing $u(x)$ for $\alpha > \beta$ is sufficient to uniquely determine the distribution of the time $\{X_t\}$ spends at or above l in the interval $[0, t]$ given that $X_0 = x$.

3. EXAMPLE 1: Sampling at Random Times

Let X_n, $n = 1, 2,\ldots$, be a sequence of independent identically distributed random variables with common distribution K and let $N(t)$, $t \geqslant 0$ be a Poisson process with intensity λ. We then define a stationary Markov process $\xi(t) = X_{N(t)}$. We will show that one can obtain $u(x)$ as defined in (2.6) explicitly for this process using the result of Corollary 2.1, and then invert the necessary transforms to determine $F_x(t, \tau)$. Here, we take $h(l) = 1$.

We note that for bounded Borel measurable f,

$$\mathscr{A}f = \lambda \left(\int_{-\infty}^{\infty} f(y)\, K\{dy\} - f \right).$$

One may easily verify that $\xi(t)$ satisfies the conditions of Corollary 2.1, so that for $\alpha > \beta$, u is the unique bounded Borel measurable solution of

$$(\alpha + \beta + \lambda)\, u(x) - \lambda \int_{-\infty}^{\infty} u(y)\, K\{dy\} = 1, \qquad \text{for} \qquad x \geqslant l,$$

and

$$(\alpha + \lambda)\, u(x) - \lambda \int_{-\infty}^{\infty} u(y)\, K\{dy\} = 1, \qquad \text{for} \qquad x < l.$$

From the above equations it is seen that

$$u(x) = \begin{cases} c_1 & x < l \\ c_2 & x \geqslant l, \end{cases}$$

where c_1 and c_2 do not depend on x. Letting

$$p = \int_{y \geqslant l} K\{dy\}$$

and $q = 1 - p$, we find that c_1 and c_2 must satisfy the linear equations

$$(\alpha + \lambda q + \beta)\, c_2 - \lambda q c_1 = 1, \qquad -\lambda p c_2 + (\alpha + \lambda p)\, c_1 = 1,$$

from which it follows that

$$c_1 = \frac{\lambda + \alpha + \beta}{\alpha(\alpha + \lambda) + \beta(\alpha + \lambda p)}$$

$$c_2 = \frac{\alpha + \lambda}{\alpha(\alpha + \lambda) + \beta(\alpha + \lambda p)}.$$

Let us consider

$$u(x) = \frac{\alpha + \lambda}{\alpha + \lambda p} \left(\frac{1}{\frac{\alpha(\alpha + \lambda)}{\alpha + \lambda p} + \beta} \right), \qquad \text{for} \qquad x \geqslant l. \tag{3.1}$$

Taking the inverse Laplace transform in (3.1) on the variable β first, we find that

$$\frac{d}{d\tau}\left(\int_0^\infty e^{-\alpha t} F(t, \tau)\, dt\right) = \frac{\alpha + \lambda}{\alpha + \lambda p} \exp\left(-\frac{\alpha(\alpha + \lambda)\tau}{\alpha + \lambda p}\right),$$

where $F = F_x$ for $x \geqslant l$. We may write the above equation as

$$\int_0^\infty e^{-\alpha t} F(t, \tau)\, dt = \frac{1}{\alpha}\left[1 - \exp\left(-\frac{\alpha(\alpha + \lambda)\,\tau}{\alpha + \lambda p}\right)\right]. \tag{3.2}$$

(Here, we use the fact that

$$\lim_{\tau \to \infty} \int_0^\infty e^{-\alpha t} F(t, \tau)\, dt = \int_0^\infty e^{-\alpha t}\, dt = \frac{1}{\alpha}.)$$

Using the fact that

$$\int_0^\infty e^{-\alpha t} I_0(2\sqrt{\mu t})\, dt = \frac{1}{\alpha} e^{\mu/\alpha}, \qquad \text{for} \qquad \mu \geqslant 0,$$

(I_0 is the modified zeroth order Bessel function), it can be easily verified that if $G(t, \tau)$ is defined by

$$G(t, \tau) = \begin{cases} 1 & \text{if} \quad t < \tau \\ \lambda q e^{-\lambda p(t-\tau)} \int_0^\tau e^{-\lambda q y} I_0(2\lambda\sqrt{pq(t-\tau)y})\, dy, & \text{if} \quad 0 \leqslant \tau \leqslant t; \end{cases} \tag{3.3}$$

then

$$\int_0^\infty e^{-\alpha t} G(t, \tau)\, dt = \frac{1}{\alpha}\left[1 - \exp\left(-\frac{\alpha(\alpha + \lambda)\,\tau}{\alpha + \lambda p}\right)\right].$$

Thus, $F(t, \tau) \equiv G(t, \tau)$, and the problem is solved.

The fact that $F(t, \tau)$ is given by (3.3) is actually equivalent to a result in reliability theory. Given that the process is below the level l we let T^+ be the time to first passage above the threshold, and similarly, if the process is initially above the level l, then we let T^- denote the time to first passage below the threshold. It follows that

$$P[T^+ \leqslant t] = 1 - \sum_{n=0}^{\infty} \frac{e^{-\lambda t}(\lambda t)^n}{n!} q^n = 1 - e^{-\lambda t}$$

and correspondingly

$$P[T^- \leqslant t] = 1 - e^{-\lambda p t}.$$

Suppose one views T^+ as the time to failure for a system and T^- as the repair time. Then the problem of finding the distribution of the total time operative during $[0, t]$ for a system with exponentially distributed time to failure with parameter λq and exponentially distributed repair time with parameter λp is equivalent to finding $F(t, \tau)$ for this process. Therefore, by Example (4) on page 80 of [6], we must have

$$F(t, \tau) = 1 - e^{-\lambda q \tau} \left[1 + \sqrt{pq\tau}\, \lambda \int_0^{t-\tau} \frac{e^{-\lambda p y}}{\sqrt{y}} I_1(2\lambda \sqrt{pq\tau y})\, dy\right]$$

$$\text{for} \quad 0 \leqslant \tau \leqslant t, \tag{3.4}$$

or by (3.3) and an integration by parts the following identity must hold

$$\lambda p e^{-\lambda q \tau} \int_0^{t-\tau} e^{-\lambda p y} I_0(2\lambda \sqrt{pq\tau y})\, dy + \lambda q e^{-\lambda p (t-\tau)} \int_0^{\tau} e^{-\lambda q y} I_0(2\lambda \sqrt{pq(t-\tau) y}) dy$$

$$+ e^{-\lambda(q\tau + p(t-\tau))} I_0(2\lambda \sqrt{pq\tau(t-\tau)}) = 1,$$

$$\text{for} \quad 0 \leqslant \tau \leqslant t.$$

The validity of (3.4) can be checked directly by taking Laplace transforms on t.

As a final remark we note that $F(t, \tau)$ may also be obtained by combinatorial methods. If τ_n is the epoch of the occurrence of the nth event in the Poisson process $N(t)$, then $\tau_1, \tau_2, \ldots, \tau_{N(t)}$ have a joint distribution that is uniform on the hyperplane $x_1 \leqslant x_2 \leqslant \cdots \leqslant x_{N(t)} \leqslant t$. A simple argument then shows that

$$F(t,\tau) = \sum_{n=0}^{\infty} P[N(t)=n] \sum_{k=0}^{n-1} \binom{n}{k} p^k q^{n-k} P[\tau_1 + \tau_2 + \cdots + \tau_{k+1} \leqslant \tau \mid N(t) = n]$$

$$= \sum_{n=0}^{\infty} \frac{e^{-\lambda t}(\lambda t)^n}{n!} \sum_{n=0}^{n-1} \binom{n}{k} p^k q^{n-k} \sum_{j=k+1}^{n} \binom{n}{j} \left(\frac{\tau}{t}\right)^j \left(1-\frac{\tau}{t}\right)$$

$$= \sum_{n=0}^{\infty} \frac{e^{-\lambda t}(\lambda t)^n}{n!} P[S_n > 0],$$

where $S_n = \sum_{k=1}^{n} Y_k$ is the n-th partial sum in a right-continuous, left-continuous random walk with

$$P[Y_1 = 1] = \frac{\tau}{t} q$$

$$P[Y_1 = -1] = \left(1 - \frac{\tau}{t}\right) p$$

$$P[Y_1 = 0] = 1 - \frac{\tau}{t} q - \left(1 - \frac{\tau}{t}\right) p.$$

A slight generalization of the known formula for the transition probabilities of a randomized Bernoulli random walk (see for example [12]) gives

$$F(t,\tau) = e^{-\lambda(\tau q + (t-\tau)p)} \sum_{n=1}^{\infty} \left(\frac{q\tau}{p(t-\tau)}\right)^{n/2} I_n(2\lambda \sqrt{pq\tau(t-\tau)}) \qquad \text{for} \quad 0 \leqslant \tau \leqslant t \tag{3.5}$$

which combined with (3.3) yields as a byproduct another identity involving modified Bessel functions of the first kind.

4. EXAMPLE 2: RANDOMIZED SIMPLE RANDOM WALK

Let $\{X_t : t \geqslant 0\}$ be randomized simple random walk. More specifically, let $\{Y_n\}$, $n = 1, 2, \ldots$, be a sequence of independent identically distributed random variables such that $P[Y_1 = 1] = P[Y_1 = -1] = \frac{1}{2}$, and let $Y_0 = x$. Let $N(t)$ be the number of events which occur during $(0, t]$ in a Poisson process with intensity λ. Then $X_t = \sum_{n=0}^{N(t)} Y_n$.

For any g in B, we have $(\mathscr{A}g)\,x = \lambda(\frac{1}{2} g(x+1) + \frac{1}{2} g(x-1) - g(x))$.

Denoting the level by n, we let $h(x) = 1$, if $x \geqslant n$ and $h(x) = 0$ otherwise. By Corollary 2.1, u as defined in (2.6) is the unique bounded Borel measurable solution of (4.1) and (4.2) below:

$$(\alpha + \beta + \lambda)\, u(x) = \tfrac{1}{2}\,\lambda[u(x+1) + u(x-1)] + 1, \quad \text{for} \quad x \geqslant n. \tag{4.1}$$

$$(\alpha + \lambda)\, u(x) = \tfrac{1}{2}\,\lambda[u(x+1) + u(x-1)] + 1, \quad \text{for} \quad x < n. \tag{4.2}$$

Solving (4.1) we have for $x \geqslant n$,

$$u(x) = \frac{1}{\alpha+\beta} + \gamma_1[\xi - \sqrt{\xi^2 - 1}]^x, \qquad \text{where} \qquad \xi = \frac{\alpha + \lambda + \beta}{\lambda}; \tag{4.3}$$

while solving (4.2) gives for $x < n$,

$$u(x) = \frac{1}{\alpha} + \gamma_2[\eta + \sqrt{\eta^2 - 1}]^x, \qquad \text{where} \qquad \eta = \frac{\alpha + \lambda}{\lambda}, \tag{4.4}$$

where γ_1 and γ_2 are constants. Setting $x = n$ in (4.1) and $x = n - 1$ in (4.2) yields two relations involving $u(n)$ and $u(n-1)$. If we use (4.3) and (4.4) in these relations, we obtain two equations for γ_1 and γ_2. Solving these equations we find that

$$\gamma_1 = \frac{-\beta(\eta - 1 + (\eta^2-1)^{1/2})}{\alpha(\alpha+\beta)\,(\xi - (\xi^2-1)^{1/2})^{n-1}\,(\xi - (\xi^2-1)^{1/2} - \eta - (\eta^2-1)^{1/2})} \tag{4.5}$$

and

$$\gamma_2 = \frac{-\beta(\xi - 1 - (\xi^2-1)^{1/2})}{\begin{pmatrix}\alpha(\alpha+\beta)\,(\eta + (\eta^2-1)^{1/2})^{n-2} \\ \times\,[(\eta + (\eta^2-1)^{1/2})\,(\xi - (\xi^2-1)^{1/2} - 2\eta) + 1]\end{pmatrix}}. \tag{4.6}$$

As $\beta \to +\infty$, $u(0)$ approaches the Laplace transform of $P[\sup_{0 \leqslant s \leqslant t} X_s < n]$. If we take $x = 0$ and $n \geqslant 1$, then using (4.4) and (4.6) one may check that

$$\begin{aligned} \lim_{\beta\to\infty} u(0) &= \frac{1}{\alpha} - \frac{1}{\alpha}\,[\eta - (\eta^2 - 1)^{1/2}]^n \\ &= \frac{1}{\alpha} - \frac{1}{\alpha}\left[\left(\frac{\alpha}{\lambda} + 1\right) - \left[\left(\frac{\alpha}{\lambda} + 1\right)^2 - 1\right]^{1/2}\right]^n. \end{aligned}$$

This last result has been shown before (see for example [13] Example 2), and one can perform the inversion to find that

$$P[\sup_{0\leqslant s\leqslant t} X_s < n] = 1 - n\int_0^t \frac{e^{-\lambda s}}{s}\, I_n(\lambda s)\, ds,$$

where I_n is the n-th order modified Bessel function of the first kind.

5. EXAMPLE 3: THE ORNSTEIN–UHLENBECK PROCESS

We now consider a special diffusion $\xi(t)$ which is stationary, Markovian and Gaussian with mean 0 and covariance $E[\xi(t_1)\,\xi(t_2)] = e^{-|t_2-t_1|}$ and with transition density $p(t, x, y)$ satisfying the backward equation

$$\frac{\partial p}{\partial t} = \frac{\sigma^2}{2}\frac{\partial^2 p}{\partial x^2} - \rho x\frac{\partial p}{\partial x}.$$

This process, also known as the Ornstein-Uhlenbeck process, describes the motion of a harmonically bound Brownian particle (with mean 0 and variance σ^2) drawn toward the origin by a force whose magnitude is proportional to its displacement with proportionality constant $\rho > 0$. The weak infinitesimal operator $\mathscr{A}$ of this process may be shown to be given by the differential operator,

$$\mathscr{A}f(x) = \frac{\sigma^2}{2}f''(x) - \rho x f'(x),$$

for functions with a continuous first derivative and a continuous second derivative at all but finitely many points. We understand $f''(x)$ to mean $[f''(x-) + f''(x+)]/2$. Equation (2.7) then becomes the parabolic cylinder equation with discontinuous coefficients

$$\frac{\sigma^2}{2}f''(x) - \rho x f'(x) - \beta h(x)f(x) - \alpha f(x) = -1,$$

where

$$h(x) = \begin{cases} 0 & \text{if} \quad x < l \\ \frac{1}{2} & \text{if} \quad x = l \\ 1 & \text{if} \quad x > l. \end{cases}$$

(It is easily checked that $h \in \tilde{B}$.)

By Corollary 2.2 we need only demonstrate the continuity of u as defined by Equation (2.4) to guarantee that it will be the unique bounded solution of the above equation. As in the proof of Theorem 2.2 we have that $u = R_\alpha(1 - \beta hu)$. Consequently, we need only show that for any bounded Borel function g, $R_\alpha g$ is continuous. However, it follows from the dominated convergence theorem that

$$\lim_{|x-y|\to 0}[R_\alpha g(x) - R_\alpha g(y)]$$

$$= \lim_{|x-y|\to 0}\int_0^\infty e^{-\alpha t}\int_0^\infty [p(t, x, z) - p(t, y, z)]\,g(z)\,dz\,dt = 0,$$

where p is the transition density of the Orstein-Uhlenbeck process defined by

$$p(t, x, y) = \frac{1}{\sqrt{2\pi}\,\sigma(t)} \exp\left[-\frac{(y - m(t))^2}{2\sigma^2(t)}\right],$$

with $m(t) = xe^{-\rho t}$ and $\sigma^2(t) = (\sigma_0^2/\rho)(1 - e^{-\rho t})$. We shall find $u(x)$ only in the case $\sigma_0^2/2 = 1$, $\rho = 1$, the general case being then easily obtainable by a change of variable. Since any solution of (5.1) below is easily seen to be in the domain of $\mathscr{A}$, the solution of (2.7) is the bounded solution of

$$u''(x) - xu'(x) - \beta h(x)\, u(x) - \alpha u(x) = -1. \tag{5.1}$$

We now state as a lemma a straightforward analogue of a result for Brownian motion proved in [4] page 56.

LEMMA. *The equation,*

$$-\frac{d}{dx}\left(-e^{-x^2/2}\frac{d}{dx}\right)u - (\beta h(x) + \alpha)\, e^{-x^2/2}u = 0 \tag{5.2}$$

(the homogeneous equation associated with (5.1) *written in self-adjoint form). has two linearly independent solutions,* $g_1 > 0$ *monotone increasing and* $g_2 > 0$ *monotone decreasing. If*

$$J(x) = e^{-x^2/2}(g_1(x)\, g_2'(x) - g_2(x)\, g_1'(x)),$$

then $J(x)$ *is a constant* J *and the Green's function for the differential operator in* (5.2) *is given by*

$$G(x, t) = \begin{cases} \dfrac{g_1(x)\, g_2(t)}{J} & \text{for} \quad x \leqslant t \\[2ex] \dfrac{g_1(t)\, g_2(x)}{J} & \text{for} \quad x > t. \end{cases}$$

In addition,

$$u(x) = \int_{-\infty}^{\infty} G(x, t)\, e^{-t^2/2}\, dt \tag{5.3}$$

is the unique bounded solution of (5.1).

Proof. The existence of the required monotonic solutions is proved exactly as in [4]. The form of the Green's function is a general result from the theory of self-adjoint second order differential operators. The proof is completed by observing that (5.3) follows from (5.1) and the definition of a Green's function.

We now determine a pair of solutions of (5.2) having the properties guaranteed by the lemma. We first remark that the Weber functions $D_{-\lambda}(x)$ and $D_{-\lambda}(-x)$, which for our purposes may be taken to be defined by

$$D_{-\lambda}(x) = \frac{1}{\Gamma(\lambda)} e^{-x^2/4} \int_0^\infty e^{-xs-\frac{1}{2}s^2} s^{\lambda-1}\, ds, \qquad \text{for} \qquad \lambda > 0,$$

are a pair of linearly independent positive solutions of Weber's equation

$$f''(x) - (\tfrac{1}{4}x^2 + \lambda - \tfrac{1}{2}) f(x) = 0.$$

It follows that $e^{x^2/4} D_{-\lambda}(x)$ and $e^{x^2/4} D_{-\lambda}(-x)$ satisfy

$$f''(x) - xf'(x) - \lambda f(x) = 0.$$

Consequently, $e^{x^2/4} D_{-(\beta+\alpha)}(x)$ and $e^{x^2/4} D_{-(\beta+\alpha)}(-x)$ satisfy (5.2) for $x \geqslant l$, while $e^{x^2/4} D_{-\alpha}(x)$ and $e^{x^2/4} D_{-\alpha}(-x)$ satisfy (5.2) for $x < l$. Since $e^{x^2/4} D_{-\lambda}(x)$ is monotone decreasing and unbounded at $-\infty$, g_1 and g_2 may be written in the following form

$$g_1(x) = \begin{cases} e^{x^2/4} D_{-\alpha}(-x) & x < l \\ A_1 e^{x^2/4} D_{-(\alpha+\beta)}(x) + B_1 e^{x^2/4} D_{-(\alpha+\beta)}(-x) & x \geqslant l, \end{cases}$$

$$g_2(x) = \begin{cases} A_2 e^{x^2/4} D_{-\alpha}(x) + B_2 e^{x^2/4} D_{-\alpha}(-x) & x < l \\ e^{x^2/4} D_{-(\alpha+\beta)}(x) & x \geqslant l. \end{cases}$$

Using the relations

$$\frac{d}{dx}[e^{x^2/4} D_{-\lambda}(\pm x)] = \mp \lambda e^{x^2/4} D_{-(\lambda+1)}(\pm x),$$

the continuity of g_1 and g_2 and their first derivatives at $x = l$ is seen to imply the following systems of linear equations for the constants A_1, A_2, B_1, and B_2.

$$A_1 D_{-(\alpha+\beta)}(l) + B_1 D_{-(\alpha+\beta)}(-l) = D_{-\alpha}(-l)$$

$$-A_1 D_{-(\alpha+\beta+1)}(l) + B_1 D_{-(\alpha+\beta+1)}(-l) = \frac{\alpha}{\alpha+\beta} D_{-(\alpha+1)}(-l)$$

and (5.4)

$$A_2 D_{-\alpha}(l) + B_2 D_{-\alpha}(-l) = D_{-(\alpha+\beta)}(l)$$

$$A_2 D_{-(\alpha+1)}(l) - B_2 D_{-(\alpha+1)}(-l) = \frac{\alpha+\beta}{\alpha} D_{-(\alpha+\beta+1)}(l).$$

We compute the Wronskian of g_1 and g_2 with the aid of the identity

$$W[D_{-\lambda}(x), D_{-\lambda}(-x)] = \lambda[D_{-\lambda}(x) D_{-(\lambda+1)}(-x) + D_{-\lambda}(-x) D_{-(\lambda+1)}(x)]$$
$$= \frac{\sqrt{2\pi}}{\Gamma(\lambda)}.$$

The result is

$$W(g_1, g_2) = -e^{-x^2/2}[\alpha D_{-(\alpha+\beta)}(l) D_{-(\alpha+1)}(-l) + (\alpha+\beta) D_{-\alpha}(-l) D_{-(\alpha+\beta+1)}(l)].$$

Consequently,

$$J(x) \equiv \alpha D_{-(\alpha+\beta)}(l) D_{-(\alpha+1)}(-l) + (\alpha+\beta) D_{-\alpha}(-l) D_{-(\alpha+\beta+1)}(l). \tag{5.5}$$

We consider first the case $x \leqslant l$. Then

$$u(x) = \frac{g_2(x)}{J} \int_{-\infty}^{x} g_1(t) e^{-t^2/2} dt + \frac{g_1(x)}{J} \int_{x}^{\infty} g_2(t) e^{-t^2/2} dt.$$

Using the identity

$$\int_x^{\infty} e^{-t^2/4} D_{-\alpha}(t) dt = e^{-x^2/4} D_{-(\alpha+1)}(x),$$

it then may be shown that

$$u(x) = \frac{A_2}{J}[D_{-\alpha}(x) D_{-(\alpha+1)}(-x) + D_{-\alpha}(-x) D_{-(\alpha+1)}(x)] + \frac{1}{J} D_{-\alpha}(-x)$$
$$\times e^{(x^2-l^2)/4}[B_2 D_{-(\alpha+1)}(-l) - A_2 D_{-(\alpha+1)}(l) + D_{-(\alpha+\beta+1)}(l)] \qquad x < l.$$

Similarly, for $x > l$ it is found that

$$u(x) = \frac{B_1}{J}[D_{-(\alpha+\beta)}(x) D_{-(\alpha+\beta+1)}(-x) + D_{-(\alpha+\beta)}(-x) D_{-(\alpha+\beta+1)}(x)]$$
$$+ \frac{1}{J} D_{-(\alpha+\beta)}(x)$$
$$\times e^{(x^2-l^2)/4}[A_1 D_{-(\alpha+\beta+1)}(l) - B_1 D_{-(\alpha+\beta+1)}(-l) + D_{-(\alpha+1)})(-l)].$$

These expressions may be simplified to yield

$$u(x) = \begin{cases} \dfrac{1}{\alpha}\left[1 - \dfrac{\beta}{J} e^{(x^2-l^2)/4} D_{-\alpha}(-x)\, D_{-(\alpha+\beta+1)}(l)\right] & x < l \\ \dfrac{1}{\alpha+\beta}\left[1 - \dfrac{\beta}{J} e^{(x^2-l^2)/4} D_{-(\alpha+\beta)}(x)\, D_{-(\alpha+1)}(-l)\right] & x \geqslant l, \end{cases} \tag{5.6}$$

where J is given by (5.5).

Letting $\beta \to \infty$ in (5.6) gives

$$\int_0^\infty e^{-\alpha t}\, dP[\xi(s) < l, 0 \leqslant s \leqslant t] = \begin{cases} \dfrac{1}{\alpha}\left[1 - \dfrac{e^{x^2/4} D_{-\alpha}(-x)}{e^{l^2/4} D_{-\alpha}(-l)}\right] & x < l \\ 0 & x \geqslant l. \end{cases} \tag{5.7}$$

This transform of the distribution of the first passage time agrees with a result of Darling and Siegert (Theorem 3.1 in [14]).

In the special case $x = l = 0$, using the relation,

$$D_{-\lambda}(0) = \frac{\sqrt{\pi}}{2^{\lambda/2}\Gamma\left(\dfrac{\lambda+1}{2}\right)},$$

we find that

$$J = \frac{\pi}{2^{(2\alpha+\beta-1)/2}} \left\{ \frac{1}{\Gamma\left(\dfrac{\alpha}{2}\right)\Gamma\left(\dfrac{\alpha+\beta+1}{2}\right)} + \frac{1}{\Gamma\left(\dfrac{\alpha+1}{2}\right)\Gamma\left(\dfrac{\alpha+\beta}{2}\right)} \right\}$$

and (5.6) then reduces to

$$u(0) = \frac{1}{\alpha+\beta}\left\{1 - \frac{\beta}{\alpha}\, \frac{1}{\dfrac{\Gamma\left(\dfrac{\alpha}{2}\right)\Gamma\left(\dfrac{\alpha+\beta+1}{2}\right)}{\Gamma\left(\dfrac{\alpha+1}{2}\right)\Gamma\left(\dfrac{\alpha+\beta}{2}\right)} + 1}\right\}.$$

An equivalent transform has been obtained by Darling and Siegert in [10].

6. EXAMPLE 4: THE COMPOUND POISSON PROCESS

We now consider compound Poisson processes. In particular, let $\{Y_n\}$, $n = 1, 2,\ldots$, be a sequence of independent identically distributed random variables, and define $Y_0 = x$. Let $N(t)$ be the number of events which occur during $(0, t]$ in a Poisson process with intensity λ. The compound Poisson process is $X_t = \Sigma_{n=0}^{N(t)} Y_n$. We have considered a special case of this process in Example 2.

By Corollary 2.1, u as defined by (2.6) is the unique bounded Borel solution of

$$(\alpha - \mathscr{A} + \beta h)\, u = 1, \tag{6.1}$$

where $h(x) = 0$ for $x < l$ and 1 for $x \geqslant l$. For a compound Poisson process

$$\mathscr{A}g = \lambda(Qg - g),$$

where

$$Qg(x) = \int_{-\infty}^{\infty} P(x, dy)\, g(y)$$

and $P(x, \Gamma) = P[Y_1 + x \in \Gamma]$, for Γ a Borel set.

For convenience, we shall assume that the distribution of Y_1 has a density function f although the arguments do not depend on this assumption.

We may write (6.1) as

$$(\alpha + \beta + \lambda)\, u(x) - \lambda \int_{-\infty}^{\infty} f(y - x)\, u(y)\, dy = 1, \qquad x \geqslant l$$

and

$$(\alpha + \lambda)\, u(x) - \lambda \int_{-\infty}^{\infty} f(y - x)\, u(y)\, dy = 1, \qquad x < l. \tag{6.2}$$

Without loss of generality (since u is a function only of $x - l$), take $l = 0$ and introduce the functions

$$u^+(x) = \begin{cases} u(x) & x \geqslant 0 \\ 0 & x < 0 \end{cases} \qquad \text{and} \qquad u^-(x) = \begin{cases} 0 & x \geqslant 0 \\ u(x) & x < 0. \end{cases}$$

One can show that for a compound Poisson process, u is a nonincreasing function of x so that $L^+ = \lim_{x\to\infty} u(x)$ and $L^- = \lim_{x\to-\infty} u(x)$ both exist. To facilitate application of the Weiner-Hopf technique we introduce the functions

$$v^+(x) = \begin{cases} u^+(x) - L^+ & x \geqslant 0 \\ -L^+ & x < 0 \end{cases} \qquad \text{and} \qquad v^-(x) = \begin{cases} -L^- & x \geqslant 0 \\ u^-(x) - L^- & x < 0. \end{cases}$$

Then (6.2) can then be written in the form

$$(\alpha+\beta+\lambda)\, v^+(x) + (\alpha+\lambda)\, v^-(x) - \lambda \int_{-\infty}^{\infty} f(y-x)\, [v^+(y) + v^-(y)]\, dy$$
$$= -(\alpha+\beta) L^+ - \alpha L^- + 1. \tag{6.3}$$

Letting x tend to $\pm\infty$ in this equation we find that

$$L^+ = \frac{1}{\alpha+\beta} \qquad \text{and} \qquad L^- = \frac{1}{\alpha},$$

which could, of course, have been shown directly from the definition of $u(x)$ as a double transform.

Now we introduce the functions

$$w^+(x) = \begin{cases} v^+(x) & x \geqslant 0 \\ 0 & x < 0 \end{cases} \qquad \text{and} \qquad w^-(x) = \begin{cases} 0 & x \geqslant 0 \\ v^-(x) & x < 0 \end{cases}$$

and note that $\lim_{x\to\pm\infty} w^\pm(x) = 0$. In terms of these functions (6.3) becomes

$$(\alpha+\beta+\lambda)\, w^+(x) + (\alpha+\lambda)\, w^-(x) - \lambda \int_{\infty}^{\infty} f(y-x)\, [w^+(y) + w^-(y)]\, dy$$
$$= \lambda\xi(x) - \lambda \int_{-\infty}^{\infty} f(y-x)\, \xi(y)\, dy, \tag{6.4}$$

where

$$\xi(x) = \begin{cases} \dfrac{1}{\alpha} & x \geqslant 0 \\[2ex] \dfrac{1}{\alpha+\beta} & x < 0. \end{cases}$$

We now proceed somewhat formally, and apply the Fourier transform to Equation (6.3). We write

$$\hat{w}^\pm(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} w^\pm(x)\, dx,$$

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} f(x)\, dx,$$

$$\hat{g}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} \left[\lambda\xi(x) - \lambda \int_{-\infty}^{\infty} f(y-x)\, \xi(y)\, dy\right] dx;$$

then, we have from (6.4)

$$[\alpha + \beta + \lambda - \sqrt{2\pi}\,\lambda\hat{f}(-k)]\,\hat{w}^+(k) + [\alpha + \lambda - \sqrt{2\pi}\,\lambda\hat{f}(-k)]\,\hat{w}^-(k) = \hat{g}(k). \tag{6.5}$$

Providing this equation holds in some strip of the k plane, $\tau_1 < \text{Im}\,(k) < \tau_2$, and the various factors have appropriate behavior at infinity, $\hat{w}^+$ and $\hat{w}^-$ may be found by the Weiner-Hopf technique.

As a particular example we take the distribution of Y_1 to be the two-sided exponential with density

$$f(x) = \tfrac{1}{2}\,\mu e^{-\mu|x|}. \tag{6.6}$$

Note that

$$\int_{-\infty}^{\infty} f(y - x)\,\xi(y)\,dy = \xi(x) - \frac{\beta}{2\alpha(\alpha + \beta)}\,e^{-\mu|x|}\,\text{sgn}(x);$$

hence,

$$\hat{g}(k) = \frac{\lambda\beta}{\sqrt{2\pi}\,\alpha(\alpha + \beta)}\left(\frac{ik}{k^2 + \mu^2}\right).$$

Since

$$\hat{f}(-k) = \frac{1}{\sqrt{2\pi}}\,\frac{\mu^2}{k^2 + \mu^2},$$

we can write (6.5) for the particular density (6.6) as

$$[(\alpha + \beta + \lambda)\,k^2 + (\alpha + \beta)\,\mu^2]\,\hat{w}^+(k) + [(\alpha + \lambda)\,k^2 + \alpha\mu^2]\,\hat{w}^-(k) = Ak, \tag{6.7}$$

where

$$A = \frac{i\lambda\beta}{\sqrt{2\pi}\,\alpha(\alpha + \beta)}. \tag{6.8}$$

In order to perform the factorization required by the Weiner-Hopf technique, we introduce the quantities

$$k_1 = \mu\sqrt{\frac{\alpha}{\alpha + \lambda}}, \qquad k_2 = \mu\sqrt{\frac{\alpha + \beta}{\alpha + \beta + \lambda}}, \tag{6.9}$$

noting that $\pm\, ik_2$ and $\pm\, ik_1$ are respectively the zeros of the coefficient of $\hat{w}^+$ and $\hat{w}^-$ in (6.7). We can then write (6.7) as

$$K^+(k)\,\hat{w}^+(k) + K^-(k)\,\hat{w}^-(k) = \frac{Ak}{(k + ik_1)\,(k - ik_2)}, \tag{6.10}$$

where

$$K^{+}(k) = \frac{(\alpha+\beta+\lambda)\,k^2 + (\alpha+\beta)\,\mu^2}{(k+ik_1)\,(k-ik_2)}$$

and

$$K^{-}(k) = \frac{(\alpha+\lambda)\,k^2 + \alpha\mu^2}{(k+ik_1)\,(k-ik_2)}. \tag{6.11}$$

Note that K^+ is analytic and nonzero in the upper half plane, Im $(k) > -k_2$ and that K^- has these properties in the lower half plane, Im $(k) < +k_2$. Now we must split the right hand side of (6.10) into the sum of a plus function and a minus function. This is done by inspection:

$$\frac{Ak}{(k+ik_1)\,(k-ik_2)} = \frac{Ck_1}{k+ik_1} + \frac{Ck_2}{k-ik_2}$$

where

$$C = \frac{A}{k_1+k_2}. \tag{6.12}$$

Hence, (6.10) can be written as

$$K^{+}(k)\,\hat{w}^{+}(k) - \frac{Ck_1}{k+ik_1} = -K^{-}(k)\,\hat{w}^{-}(k) + \frac{Ck_2}{k-ik_2}. \tag{6.13}$$

We assume sufficiently restrictive behavior for w^+ and w^-, so that (6.13) holds in a strip of the complex k plane, $\tau_1 < \text{Im}\,(k) < \tau_2$ (exponentially decreasing at $\pm\,\infty$ will more than suffice). The usual analytic continuation argument shows that (6.13) defines a function analytic throughout the finite k plane. Since u itself is the double transform of a probability distribution, it is certainly bounded, and hence so are w^+ and w^-. Thus, $\hat{w}^{\pm}(k) = O(1/k)$ for large $|\,k\,|$. Since $K^{\pm}$ are bounded as $|\,k\,| \to \infty$, the function defined by (6.13) must be identically zero (Liouville's theorem). We deduce from (6.13) that

$$\hat{w}^{+}(k) = \frac{Ck_1}{\alpha+\beta+\lambda}\,\frac{1}{k+ik_2}$$

and 6.14)

$$\hat{w}^{-}(k) = \frac{Ck_2}{\alpha+\lambda}\,\frac{1}{k-ik_1}.$$

Applying the Fourier inversion formula, we find that

$$w^+(x) = -\frac{\sqrt{2\pi}\, iCk_1}{\alpha+\beta+\lambda} e^{-k_2 x}$$

$$w^-(x) = \frac{\sqrt{2\pi}\, iCk_2}{\alpha+\lambda} e^{k_1 x}.$$

Thus w^+ and w^- approach 0 exponentially as $|x|$ approaches infinity. Arguing in reverse, one may recover (6.13) which is equivalent to (6.4). Consequently, w^+ and w^- are solutions of (6.4).

This gives for u, the solution of (6.2) with f given by (6.6),

$$u(x) = \begin{cases} \dfrac{1}{\alpha+\beta}\left[1 + \dfrac{\lambda\beta k_1}{\alpha(\alpha+\beta+\lambda)(k_1+k_2)} e^{-k_2(x-l)}\right] & x \geqslant l \\ \dfrac{1}{\alpha}\left[1 - \dfrac{\lambda\beta k_2}{(\alpha+\beta)(\alpha+\lambda)(k_1+k_2)} e^{k_1(x-l)}\right] & x < l. \end{cases} \tag{6.15}$$

As $\beta \to \infty$ we obtain the Laplace transform of the first passage time. Using (6.15) we find that

$$\lim_{\beta\to\infty} u(x) = \int_0^\infty e^{-\alpha t} P_x[X(s) < l, 0 \leqslant s \leqslant t]\, dt$$

is given by

$$\lim_{\beta\to\infty} u(x) = \begin{cases} 0 & x \geqslant l \\ \dfrac{1}{\alpha}\left[1 - \left(1 - \sqrt{\dfrac{\alpha}{\alpha+\lambda}}\right) \exp\left\{\sqrt{\dfrac{\alpha}{\alpha+\lambda}}(x-l)\right\}\right] & x < l. \end{cases} \tag{6.16}$$

From (6.16) we deduce the result

$$\lim_{x\uparrow l} \lim_{\beta\to\infty} u(x) = \int_0^\infty e^{-\alpha t} P_l[X(s) \leqslant l, 0 \leqslant s \leqslant t]\, dt = \frac{1}{\sqrt{\alpha(\alpha+\lambda)}},$$

which may be verified using the fluctuation theory associated with random walk (see [12]). The inversion of this Laplace transform gives

$$P_l[X(s) \leqslant l, 0 \leqslant s \leqslant t] = e^{-(\lambda/2)t} I_0\left(\frac{\lambda}{2} t\right).$$

It is of some interest to note that when $x < l$, the exponential function in the solution for u does not involve β; in this case we can invert the transform on β. Substituting the expressions given by (6.9) for k_1 and k_2 in the second formula of (6.15) we find that for $x < l$,

$$u(x) = \frac{1}{\alpha}\left\{1 + \left[\sqrt{\frac{\alpha(\alpha+\beta+\lambda)}{(\alpha+\lambda)(\alpha+\beta)}} - 1\right] e^{k_1(x-l)}\right\}$$

$$= \frac{1}{\alpha} - \frac{1}{\alpha}\left[1 - \sqrt{\frac{\alpha}{\alpha+\lambda}}\right] e^{k_1(x-l)} + \frac{1}{\alpha}\sqrt{\frac{\alpha}{\alpha+\lambda}}\left[\sqrt{\frac{\alpha+\beta+\lambda}{\alpha+\beta}} - 1\right] \times e^{k_1(x-l)}.$$

In this latter form it is straightforward to invert on β to obtain

$$\int_0^\infty e^{-\alpha t} P_x[H(t) \leqslant \tau]\, dt = \frac{1}{\alpha}\left\{1 - \left(1 - \sqrt{\frac{\alpha}{\alpha+\lambda}}\right) e^{k_1(x-l)}\right\} + \frac{\lambda}{2}\exp\left[k_1(x-l) - \left(\alpha + \frac{\lambda}{2}\right)\right] \times \sqrt{\frac{\alpha}{\alpha+\lambda}} \int_0^\tau \left[I_0\left(\frac{\lambda t}{2}\right) + I_1\left(\frac{\lambda t}{2}\right)\right] dt$$

for $x < l$.

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