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Distribution of Time Above a Threshold for semi-Markov Jump Processes*

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1. INTRODUCTION

In this paper we extend to semi-Markov jump processes the investigation of the distribution of time above a threshold which was made by Stone, Belkin, and Snyder in [1]. A semi-Markov jump process is a generalization to continuous state spaces of a semi-Markov process, as defined by Pyke and Shaufele [2] or Yackel [3]. Semi-Markov jump processes have been defined previously by Stone [4]. This definition is repeated in Section 2.

Let $\{X_t : t \geqslant 0\}$ be a semi-Markov jump process and $H_\ell(t)$ be the amount of time the process spends at or above the level ℓ in the interval $[0, t]$. We are primarily interested in finding $F_x(t, \tau) = P_{x,0}[H_\ell(t) \leqslant \tau]$, where $P_{x,0}$ denotes a probability conditioned on the process having just entered x at time 0. We attack the problem of finding F_x by finding an integral equation [i.e., (2.9)] whose unique solution, for a wide class of processes, is the transform u given by

$$u(x, \alpha) = \int_0^\infty \int_0^\infty e^{-\alpha t - \beta \tau} F_x(t, d\tau)\, dt.$$

We consider the problem of finding the distribution of time above a threshold as a special case of finding the distribution of functionals of the form

$$H(t) = \int_0^t h(X_s)\, ds$$

where h is a nonnegative bounded Borel function and $\{X_t : t \geqslant 0\}$ is a semi-Markov jump process. In the case where $h(x) = 0$ for $x < \ell$ and 1 for $x \geqslant \ell$, H becomes H_ℓ. We examine

$$u_f(x, \alpha) = \int_0^\infty e^{-\alpha t} E_{x,0}[e^{-\beta H(t)} f(X_t)]\, dt,$$

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where f is a bounded Borel function and $E_{x,0}$ denotes expectation conditioned on the process having just entered state x at time 0. In Section 2 we show that for a semi-Markov jump process, u_f is a solution of the integral Eq. (2.7). Conditions are found which guarantee that u_f is the unique bounded Borel solution of (2.7).

In Section 3 we find u for a class of examples which may be thought of as sampling at renewal times. When the inter-renewal times have gamma distributions with integer order j, we find the distribution of the first passage time to or above the level ℓ. When $j = 2$, we are able to invert u with respect to β to obtain

$$G(\tau, \alpha) = \int_0^\infty e^{-\alpha t} P[H_\ell(t) \leqslant \tau \mid X_0 < \ell]\, dt. \tag{1.1}$$

Define the stopping variable T_τ as follows:

$$T_\tau = \inf\{t : H_\ell(t) > \tau\}.$$

Using $G(\tau, \alpha)$, we are able to find the transform of T_τ and then $E[T_\tau]$ for the case $j = 2$. This case is of interest since it appears that there is some empirical support for using this process as a model for acoustic fluctuations in the ocean (see Chap. IV of [5]).

In Section 4 we find u for a class of processes in which the time between increments has a gamma distribution and the increments have a two-sided symmetric exponential distribution. By taking a limit as $\beta \to \infty$, we are able to find the transform of the time of the first passage to or above ℓ. In a special case, we are able to perform the inversion to find the first passage time distribution explicitly.

The previous investigations of the distribution of $H(t)$ have concentrated on Markov processes. In the case of a Wiener process, $H(t)$ is sometimes referred to as Kac's Brownian functional (see Ito and MacKean [6]) because of Kac's work described in [7]. Beekman [8] investigated the distribution of $H(t)$ for Gaussian-Markov processes. For a measurable Markov process with stationary transition probabilities, the distribution of $H(t)$ has been studied by Darling and Siegert [9], and Stone, Belkin, and Snyder [1]. In the latter paper, the double transform of $H(t)$ was computed for several special processes and inverted for the case of the process of section 3 with $j = 1$.

For non-Markovian processes, Leadbetter and Cryer [10] have investigated the moments of the time spent above continuous curves for stationary, normal processes. The standard fluctuation theory for random walks has been extended to renewal-reward processes by Jewell [11]. (Renewal-reward processes are the special class of semi-Markov jump processes described in Section 4 of this paper.) For a certain class of renewal-reward processes Jewell shows that $H_0(t)/t$ has an arcsine distribution in the limit as $t \to \infty$.

2. An Integral Equation for u_f

In this section we find an integral equation, namely (2.7), which is satisfied by u_f. To do this, we first prove Theorem 2.1 which provides the basic relation needed to derive (2.7). Corollaries 2.1 and 2.2 translate Theorem 2.1 into operator Eqs. (2.4) and (2.5) [(2.5) being a special case of (2.4)] and give conditions which guarantee the uniqueness of the solutions of (2.4) and (2.5). Corollaries 2.3 and 2.4 apply the results of Corollaries 2.1 and 2.2 to find Eqs. (2.7) and (2.8) [(2.8) being a special case of (2.7)] which are satisfied by u_f and to find conditions under which u_f is the unique solution of (2.7) or (2.8). Before stating and proving Theorem 2.1, we make some definitions.

Semi-Markov jump processes have been defined previously in [4]. However, we reproduce that definition here for the convenience of the reader.

Let $\{X_t : t \geqslant 0\}$ be a separable stochastic process with random variables, X_t, defined on the probability space $(\Omega, \mathfrak{A}, P)$, and having their range in R, the real numbers. Define

$$Y_t = \begin{cases} t & \text{if} \quad X_s = X_t \quad \text{for all} \quad 0 \leqslant s \leqslant t, \\ t - \sup\{s : 0 \leqslant s \leqslant t, \text{ and } X_s \neq X_t\} & \text{otherwise,} \end{cases}$$

and

$$\nu_t = \inf\{s : s > t \text{ and } X_s \neq X_t\} \quad \text{for} \quad t \geqslant 0.$$

For convenience of notation, let $\nu = \nu_0$.

Let $R^+ = [0, \infty)$; $\mathscr{B}(R)$ and $\mathscr{B}(R^+)$ denote the σ algebras of Borel subsets of R and R^+, respectively. Define a two-dimensional process $\{(X_t, Y_t) : t \geqslant 0\}$; the state space of this process is a subset of $R \times R^+$. If Π is a member of $\mathscr{B}(R) \times \mathscr{B}(R^+)$, then for (x, y) in $R \times R^+$, let $P_{x,y}[(X_t, Y_t) \in \Pi]$ be a version of the conditional probability $P[(X_t, Y_t) \in \Pi \mid (X_0, Y_0) = (x, y)]$.

We say that $\{X_t : t \geqslant 0\}$ is a *semi-Markov jump process* if the two-dimensional process $\{(X_t, Y_t) : t \geqslant 0\}$ has right-continuous sample paths, is a strong Markov process with stationary transition probabilities, and

$$P_{x,0}[0 < \nu < \infty] = 1$$

for all x in R.

The intent of the condition on ν is to guarantee that (with probability 1) the process remains a positive length of time in each state it enters (the finiteness of ν is not essential, and with appropriate modifications ν could be allowed to be infinite).

We wish to guarantee that ν is a stopping time of the $\{(X_t, Y_t)\}$ process so that we may apply the strong Markov property to ν. In order to assure that ν is a stopping time, it is necessary to guarantee that for each t, $X_{\nu_t} \neq X_t$, i.e., that jump points cannot accumulate from the right to a point at which no jump occurs. It is for this reason that we require the $\{Y_t\}$ process to be right continuous. The $\{X_t\}$ process then moves strictly by jumps with ν_t, the time to the first jump following time t, and X_{ν_t} the state to which the process jumps.

Since ν is a stopping time and $\{X_t\}$ is right continuous, X_ν is a random variable, and we may define a function q on $R \times \mathscr{B}(R) \times R^+$ by

$$q(x, \Gamma, t) = P_{x,0}\{X_\nu \in \Gamma \text{ and } \nu \leqslant t\}.$$

Note that

$$q(x, \Gamma, t) = \int_\Gamma \int_{[0,t]} c(x, z, ds)\, k(x, dz),$$

where

$$c(x, z, s) = P_{x,0}[\nu \leqslant s \mid X_\nu = z] \qquad \text{and} \qquad k(x, \Gamma) = P_{x,0}[X_\nu \in \Gamma].$$

Define

$$a(x, t) = q(x, R, t) = P_{x,0}[\nu \leqslant t].$$

The condition that ν be positive and finite may be expressed as $a(x, 0) = 0$ and $a(x, t) \to 1$ as $t \to \infty$ for all x in R.

The right-continuity of $\{(X_t, Y_t)\}$ implies that it is a measurable process; thus, for a fixed set Π, the transition functions are Borel measurable.

If the state space of a semi-Markov jump process is countable, then the process becomes a special case of a semi-Markov process as defined in [2] and [3]. In this case the function $q(i, j, \cdot) - q(i, j - 1, \cdot)$ becomes the function $Q_{ij}(\cdot)$ of [2]. If $c(x, y, t) = 1 - e^{-\gamma(x)t} (0 < \gamma(x) < \infty)$, then $q(x, \Gamma, t) = (1 - e^{-\gamma(x)t})\, k(x, \Gamma)$, and the semi-Markov jump process becomes a Markov jump process as described in Feller [12], p. 316.

In this paper we take the point of view that the functions a and q are known and that the distributions of functionals are to be solved in terms of them.

Let h be a nonnegative Borel function and for ω in Ω define

$$H(t, \omega) = \int_0^t h[X_s(\omega)]\, ds.$$

Note that $H(t)$ is well defined because the sample paths of the $\{X_t\}$ process are Borel functions. For each ω in Ω, let ω_s^+ be such that

$$[X_t(\omega_s^+), Y_t(\omega_s^+)] = [X_{t+s}(\omega), Y_{t+s}(\omega)].$$

Observe that

$$H(t, \omega) = H(s, \omega) + H(t - s, \omega_s^+).$$

In the case where $h(x) = 1$ for $x \geqslant \ell$ and 0 for $x < \ell$, we shall designate H by H_ℓ. Then $H_\ell(t)$ gives the amount of time the process $\{X_t\}$ spends at or above the level ℓ.

For fixed $\beta > 0$, let v be defined on $R \times \mathscr{B}(R) \times R^+$ by

$$v(x, \Gamma, t) = E_{x,0}[e^{-\beta H(t)} \chi_\Gamma(X_t)],$$

where χ_Γ is the characteristic function of the set Γ. The measurability of $\{(X_t, Y_t)\}$ guarantees that $v(\cdot, \Gamma, \cdot)$ is Borel measurable. The following is the basic theorem of this paper.

THEOREM 2.1. *For a semi-Markov jump process,*

$$v(x, \Gamma, t) = [1 - a(x, t)]\, e^{-\beta h(x) t} \chi_\Gamma(x) + \int_0^t \int_R v(z, \Gamma, t - s)\, e^{-\beta h(x) s} q(x, dz, ds). \tag{2.1}$$

Proof. We have

$$E_{x,0}\{e^{-\beta H(t)} \chi_\Gamma(X_t); [\nu > t]\} = e^{-\beta h(x) t} \chi_\Gamma(x)\, [1 - a(x, t)], \tag{2.2}$$

where $E_{x,0}\{Z; [\nu > t]\}$ indicates that the expectation of the random variable Z is to be taken over the set $[\nu > t]$. A similar comment applies below, and

$$\begin{aligned} E_{x,0}&[e^{-\beta H(t)} \chi_\Gamma(X_t); [\nu \leqslant t]] \\ &= E_{x,0}[e^{-\beta h(x)\nu} e^{-\beta H(t-\nu, \omega_\nu^+)} \chi_\Gamma(X_{t-\nu}(\omega_\nu^+)); [\nu \leqslant t]] \\ &= E_{x,0}[\chi_{[\nu \leqslant t]} e^{-\beta h(x)\nu} E_{x_\nu,0}[e^{-\beta H(t-\nu, \omega_\nu^+)} \chi_\Gamma(X_{t-\nu}(\omega_\nu^+))]] \qquad (2.3) \\ &= E_{x,0}[e^{-\beta h(x)\nu} v(X_\nu, \Gamma, t - \nu); [\nu \leqslant t]] \\ &= \int_0^t \int_R e^{-\beta h(x) s} v(z, \Gamma, t - s)\, q(x, dz, ds), \end{aligned}$$

where the third equality follows from the strong Markov property applied to ν. Using the definition of v and combining (2.2) with (2.3), we obtain (2.1), and the theorem is proved.

Before stating the next corollary, we must define some operators. Let w be a real valued function defined on $R \times \mathscr{B}(R)$ such that

(i) $w(\cdot, \Gamma)$ is Borel measurable for each Γ in $\mathscr{B}(R)$

(ii) $w(x, \cdot)$ is a positive finite measure on $\mathscr{B}(R)$ for each x in R.

Then w is called a kernel function, and it defines an operator W which maps

B, the space of bounded Borel functions defined on R, into itself as follows: If f is in B, then

$$Wf(x) = \int_R f(y)\, w(x, dy).$$

We consider the space B with the supremum norm. Thus, the operator W has norm

$$\| W \| = \sup_x w(x, R).$$

Let $V^*(\alpha)$ and $Q^*(\alpha)$ be the operators on B with kernel functions v^* and q^* defined by

$$v^*(x, \Gamma, \alpha) = \int_0^\infty e^{-\alpha t} v(x, \Gamma, t)\, dt$$

and

$$q^*(x, \Gamma, \alpha) = \int_0^\infty e^{-[\alpha+\beta h(x)]t} q(x, \Gamma, dt) = \int_\Gamma \int_0^\infty e^{-[\alpha+\beta h(x)]t} c(x, z, dt)\, k(x, dz)$$

Note that for $\alpha > 0$,

$$\| V^*(\alpha) \| \leqslant \frac{1}{\alpha}, \qquad \text{and} \qquad \| Q^*(\alpha) \| \leqslant 1.$$

For $\alpha > 0$, let

$$a^*(x, \alpha) = \int_0^\infty e^{-\alpha t} a(x, dt),$$

and let $B^*(\alpha)$ be the operator with kernel function b^* given by

$$b^*(x, \Gamma, \alpha) = \frac{1 - a^*[x, \alpha + \beta h(x)]}{\alpha + \beta h(x)} \chi_\Gamma(x).$$

For $\alpha > 0$, $\| B^*(\alpha) \| < 1/\alpha$.

COROLLARY 2.1. *For a semi-Markov jump process*

$$V^*(\alpha) = B^*(\alpha) + Q^*(\alpha)\, V^*(\alpha) \qquad \text{for} \qquad \alpha > 0. \tag{2.4}$$

Moreover, if $\| Q^*(\alpha_0) \| < 1$ *for some* α_0, *then for* $\alpha \geqslant \alpha_0$, $V^*(\alpha)$ *is the unique bounded solution of* (2.4).

Proof. Equation (2.4) follows immediately from (2.1) upon taking Laplace transforms and observing that the operator $Q^*(\alpha)\, V^*(\alpha)$ has kernel function given by

$$\int_R v^*(z, \Gamma, \alpha)\, q^*(x, dz, \alpha).$$

Thus, $V^*(\alpha)$ is a bounded solution of (2.4). Suppose that $\| Q^*(\alpha_0)\| < 1$; then for $\alpha \geqslant \alpha_0$, $\| Q^*(\alpha)\| < 1$, and we may write

$$V^*(\alpha) = (1 - Q^*(\alpha))^{-1}\, B^*(\alpha)$$

proving that $V^*(\alpha)$ is the unique bounded solution of (2.4). The corollary follows.

Let us consider the case where ν is stochastically independent of X_ν. Then for each x and t, $c(x, \cdot, t)$ is a constant function and one may easily verify that

$$q(x, \Gamma, t) = \int_\Gamma \int_0^t a(x, ds)\, k(x, dz).$$

Let $A^*(\alpha)$ be the operator with kernel function d given by

$$d(x, \Gamma, \alpha) = a^*[x, \alpha + \beta h(x)]\, \chi_\Gamma(x),$$

and let K be the operator with kernel function k. Then it follows that

$$Q^*(\alpha) = A^*(\alpha)\, K.$$

Corollary 2.2. *For a semi-Markov process in which* ν *is independent of* X_ν

$$V^*(\alpha) = B^*(\alpha) + A^*(\alpha)\, K V^*(\alpha) \qquad \textit{for} \qquad \alpha > 0. \tag{2.5}$$

Moreover, if for some α_0,

$$\sup_x a^*(x, \alpha) < 1, \tag{2.6}$$

then $V^*(\alpha)$ *is the unique bounded solution of* (2.5) *for* $\alpha \geqslant \alpha_0$.

Proof. That $V^*(\alpha)$ is a bounded solution of (2.5) follows from Corollary 2.1 and the fact that if ν is independent of X_ν, then $Q^*(\alpha) = A^*(\alpha)\, K$. If (2.6) holds, then because $\beta > 0$ and $h \geqslant 0$, $\| A^*(\alpha)\| < 1$ for all $\alpha \geqslant \alpha_0$. Since $\| K \| = 1$ and since

$$\| Q^*(\alpha)\| \leqslant \| A^*(\alpha)\|\, \| K \|,$$

we have that $\| Q^*(\alpha)\| < 1$ for all $\alpha \geqslant \alpha_0$. Then by Corollary 2.1 $V^*(\alpha)$ is the unique bounded solution of (2.5), and the corollary is proved.

If we apply $V^*(\alpha)$ to a bounded Borel function f, and define $u_f(x, \alpha) = V(\alpha) f(x)$, we have

$$u_f(x, \alpha) = \int_0^\infty e^{-\alpha t} E_{x,0}[e^{-\beta H(t)} f(X_t)\, dt].$$

COROLLARY 2.3. *For a semi-Markov jump process,*

$$\sup_x q^*(x, R, \alpha_0) < 1$$

for some $\alpha_0 > 0$, *implies that* $u_f(\cdot, \alpha)$ *is the unique bounded Borel solution of*

$$u_f(x, \alpha) = \frac{1 - a^*(x, \alpha + \beta h(x))}{\alpha + \beta h(x)} f(x) + \int_R u_f(z, \alpha)\, q^*(x, dz, \alpha) \tag{2.7}$$

for all $\alpha > \alpha_0$.

Proof. The corollary follows from Corollary 2.2 and applying both sides of (2.4) to the function f.

COROLLARY 2.4. *For a semi-Markov jump process, in which* v *is independent* X_v, *we have*

$$\sup_x a^*(x, \alpha) < 1$$

for some $\alpha_0 > 0$ *implies that* $u_f(\cdot, \alpha)$ *is the unique bounded Borel solution of*

$$u_f(x, \alpha) = \frac{1 - a^*[x, \alpha + \beta h(x)]}{\alpha + \beta h(x)} f(x) + a^*[x, \alpha + \beta h(x)] \int_R u_f(z, \alpha)\, k(x, dz)$$

for all $\alpha > \alpha_0$.

Proof. The corollary follows from Corollary 2.2 and applying both sides of Eq. (2.5) to the function f.

In the case where $f = 1$ and $H = H_\ell$, we shall designate u_f simply by u. Then

$$u(x, \alpha) = \int_0^\infty e^{-\alpha t} \int_0^\infty e^{-\beta\tau} F_x(t, d\tau)\, dt,$$

where

$$F_x(t, \tau) = P_{x,0}[H_\ell(t) \leqslant \tau].$$

That is, $u(x, \alpha)$ is the double transform of the amount of time the process $\{X_t\}$ spends at or above the threshold ℓ during the interval $[0, t]$ given that the process has just entered the state x at time 0.

Under the conditions of Corollary 2.3, u is the unique bounded Borel solution of

$$u(x, \alpha) = \frac{1 - a^*(x, \alpha + \beta h(x))}{\alpha + \beta h(x)} + \int_R u(z, \alpha)\, q^*(x, dz, \alpha), \tag{2.9}$$

and under the conditions of Corollary 2.4, u is the unique bounded Borel solution of

$$u(x, \alpha) = \frac{1 - a^*[x, \alpha + \beta h(x)]}{\alpha + \beta h(x)} + a^*[x, \alpha + \beta h(x)] \int_R u(z, \alpha)\, k(x, dz). \tag{2.10}$$

We note that in principal one can calculate $P_{x,y}[H_\ell(t) \leqslant \tau]$ from F_x as follows:

$$P_{x,y}[H_\ell(t) \leqslant \tau] = \begin{cases} \int_0^t \int_R F_z(t - s, \tau)\, q_y(x, dz, ds) & \text{for} \quad x < \ell, \\ \int_0^t \int_R F_z(t - s, \tau - s)\, q_y(x, dz, ds) & \text{for} \quad x \geqslant \ell, \end{cases}$$

where

$$q_y(x, \Gamma, s) = P_{x,y}[X_\nu \in \Gamma \text{ and } \nu \leqslant s].$$

3. Example 1: Sampling at Renewal Times

In this example we find u, the double transform of the amount of time spent at or above the level ℓ, for a process which may be thought of as sampling at renewal times. Consider a renewal process in which the time between renewals has distribution function a. Let $N(t)$ equal the number of renewals which have occurred in the interval $(0, t]$. We assume that inter-renewal times are strictly positive and finite with probability 1, that a renewal occurred at time 0, and that $N(t)$ is right continuous. Let $\{Z_n\}$, $n = 0, 1,\ldots$, be a sequence of independent identically distributed random variables with distribution function k such that $\{Z_n\}$ is independent of $\{N(t)\}$. Finally, let $X_t = Z_{N(t)}$.

One may easily verify that $\{X_t : t \geqslant 0\}$ is a semi-Markov jump process and that X_ν is independent of ν so that

$$q(x, \Gamma, t) = \int_0^t \int_\Gamma k(dy)\, a(ds).$$

By Corollary 2.4, u is the unique bounded Borel solution of (2.9) which becomes

$$u(x, \alpha) = \frac{1 - a^*(\alpha + \beta)}{\alpha + \beta} + a^*(\alpha + \beta) \int_R u(y, \alpha)\, k(dy) \qquad \text{for} \qquad x \geqslant \ell \tag{3.1}$$

and

$$u(x, \alpha) = \frac{1 - a^*(\alpha)}{\alpha} + a^*(\alpha) \int_R u(y, \alpha)\, k(dy) \qquad \text{for} \qquad x < \ell, \tag{3.2}$$

where

$$a^*(\alpha) = \int_0^\infty e^{-\alpha t} a(dt).$$

From (3.1) and (3.2), one can see that

$$u(x, \alpha) = \begin{cases} c_1 & x < \ell \\ c_2 & x \geqslant \ell, \end{cases}$$

where c_1 and c_2 do not depend on x. Let $p = P[Z_1 \geqslant \ell]$ and $\bar{p} = 1 - p$, then we obtain

$$c_1 = \frac{1 - a^*(\alpha)}{\alpha} + a^*(\alpha)\,(\bar{p}c_1 + pc_2)$$

$$c_2 = \frac{1 - a^*(\alpha + \beta)}{\alpha + \beta} + a^*(\alpha + \beta)\,(\bar{p}c_1 + pc_2).$$

Solving for c_1 and c_2, we have

$$c_1 = \frac{(\alpha + \beta)\,[1 - a^*(\alpha)]\,[1 - pa^*(\alpha + \beta)] + \alpha p a^*(\alpha)\,[1 - a^*(\alpha + \beta)]}{\alpha(\alpha + \beta)\,[1 - \bar{p}a^*(\alpha) - pa^*(\alpha + \beta)]} \tag{3.3}$$

and

$$c_2 = \frac{\alpha[1 - \bar{p}a^*(\alpha)]\,[1 - a^*(\alpha + \beta)] + (\alpha + \beta)\,\bar{p}a^*(\alpha + \beta)\,[1 - a^*(\alpha)]}{\alpha(\alpha + \beta)\,[1 - \bar{p}a^*(\alpha) - pa^*(\alpha + \beta)]}. \tag{3.4}$$

We now consider the class of processes for which

$$a^*(\alpha) = \left(\frac{\lambda}{\lambda + \alpha}\right)^j, \tag{3.5}$$

where j is a fixed positive integer. (The case $j = 1$ has been considered in [1] where u was inverted to find F_x.) We now find the distribution of the first passage time to the level ℓ for this class of processes.

Define

$$M_x(t) = P_{x,0}[X_s < \ell : 0 \leqslant s \leqslant t]; \tag{3.6}$$

then

$$\lim_{\beta\to\infty} u(x, \alpha) = \int_0^\infty e^{-\alpha t} M_x(t)\, dt. \tag{3.7}$$

Using (3.3), (3.5), (3.7), and letting $\beta \to \infty$, one finds that

$$\int_0^\infty e^{-\alpha t} M_x(t)\, dt = \begin{cases} \dfrac{1}{\alpha}\left[1 - \dfrac{p\lambda^j}{(\lambda+\alpha)^j - \bar{p}\lambda^j}\right] & \text{for} \quad x < \ell \\ 0 & \text{for} \quad x \geqslant \ell. \end{cases} \tag{3.8}$$

Let M_x' denote the derivative of M_x. Then inverting (3.8), we find that for $x < \ell$,

$$M_x'(t) = \begin{cases} \dfrac{rp}{j\bar{p}} e^{-\lambda t} \left\{ e^{rt} + 2 \sum_{n=1}^{(j-1)/2} \exp\left[rt \cos\left(\dfrac{2\pi n}{j}\right)\right] \times \cos\left[rt \sin\left(\dfrac{2\pi n}{j}\right) + \dfrac{2\pi n}{j}\right] \right\} & \text{for} \quad j \text{ odd} \\ \dfrac{rp}{j\bar{p}} e^{-\lambda t} \left\{ 2\sinh(rt) + 2 \sum_{n=1}^{(j-2)/2} \exp\left[rt \cos\left(\dfrac{2\pi n}{j}\right)\right] \times \cos\left[rt \sin\left(\dfrac{2\pi n}{j}\right) + \dfrac{2\pi n}{j}\right] \right\} & \text{for} \quad j \text{ even,} \end{cases}$$

where $r = \lambda q^{1/j}$.

For the rest of this example we fix $j = 2$ and find the transform of the first passage time of $\{H_\ell(t) : t \geqslant 0\}$ through the level τ, given that $X_0 < \ell$. Define the random variable

$$T_\tau = \inf\{t : H_\ell(t) > \tau\}.$$

Because $H_\ell(t)$ is nondecreasing, the set $[T_\tau < t] = [H(t) > \tau]$. Let

$$\varphi(\alpha) = \int_0^\infty e^{-\alpha t}\, dP[T_\tau < t \mid X_0 < \ell].$$

Then recalling the definition of $G(\tau, \alpha)$ given in (1.1), we have that

$G(\tau, \alpha) = [1 - \varphi(\alpha)]/\alpha$. We now invert c_1/β with respect to β, to obtain G for case $j = 2$. From (3.3) and (3.5), we have

$$\frac{1}{\beta} c_1 = \frac{(2\lambda + \alpha)(\lambda + \alpha + \beta)^2 + p\beta\lambda^2}{\beta\{[(\lambda + \alpha)^2 - \bar{p}\lambda^2](\alpha + \beta + \lambda)^2 - p\lambda^2(\alpha + \lambda)^2\}}$$

$$= \frac{1}{\alpha}\left[\frac{1}{\beta} + \frac{B_1}{\beta + (\alpha + \lambda)\left(1 + \lambda\sqrt{\frac{p}{A}}\right)} + \frac{B_2}{\beta - (\alpha + \lambda)\left(-1 + \lambda\sqrt{\frac{p}{A}}\right)}\right]$$

where $A = (\lambda + \alpha)^2 - \bar{p}\lambda^2$,

$$B_1 = \frac{-\lambda^2[-\sqrt{pA} + (\alpha + \lambda)p]}{2(\alpha + \lambda)A}$$

and

$$B_2 = \frac{-\lambda^2[-\sqrt{pA} + (\alpha + \lambda)p]}{2(\alpha + \lambda)A}.$$

Thus,

$$G(\tau, \alpha) = \frac{1}{\alpha}\left\{1 - B_1 \exp\left[-\tau(\alpha + \lambda)\left(1 + \lambda\sqrt{\frac{p}{A}}\right)\right] - B_2 \exp\left[\tau(\alpha + \lambda)\left(-1 + \lambda\sqrt{\frac{p}{A}}\right)\right]\right\}$$

and

$$\varphi(\alpha) = \frac{\lambda^2}{(\alpha + \lambda)^2} \exp[-\tau(\alpha + \lambda)]\,[z \sinh(\lambda\tau z) + z^2 \cosh(\lambda\tau z)],$$

where

$$z = (\alpha + \lambda)p^{\frac{1}{2}}[(\alpha + \lambda)^2 - \bar{p}\lambda^2]^{-\frac{1}{2}}.$$

Since $-\varphi'(0) = E[T_\tau]$, we have

$$E[T_\tau] = \frac{1}{\lambda p}\left[\lambda\tau + 1 + p + \frac{\bar{p}}{2}(1 + e^{-2\tau\lambda})\right],$$

the expected time to accumulate the amount of time τ at or above the level ℓ given that $X_0 < \ell$.

4. Example 2: Compound Renewal Processes

In this example we find u, the double transform of the amount of time spent at or above the level ℓ, for a process in which sums of independent identically distributed random variables accumulate at renewal times. In particular, let $N(t)$ be the number of renewals which occur in $(0, t]$ in a renewal process in which a renewal has occurred at time 0. Let $\{Z_i\}$, $i = 0, 1, 2,...$, be a sequence of independent identically distributed random variables with the sequence $\{Z_n\}$ independent of $N(t)$. Define

$$X_t = \sum_{i=0}^{N(t)} Z_i .$$

We call $\{X_t : t \geqslant 0\}$ a compound renewal process. When conditioned on $X_0 = 0$, $\{X_t\}$ becomes a renewal-reward process as defined in [11].

For this example, we suppose that Z_i has a two-sided exponential distribution with density $\frac{1}{2}\mu \exp(-\mu \mid x \mid)$, and that the inter-renewal times have a gamma distribution with probability density

$$a'(t) = \frac{1}{\Gamma(n)} \lambda^n t^{n-1} e^{-\lambda t},$$

where n is a fixed positive number.

One may easily verify that $\{X_t\}$ is a semi-Markov jump process and that X_ν is independent of ν. By Corollary 2.4, u is the unique bounded Borel solution of

$$u(x, \alpha) = \frac{1 - \left(\dfrac{\lambda}{\alpha + \lambda + \beta h(x)}\right)^n}{\alpha + \beta h(x)} + \left(\frac{\lambda}{\lambda + \alpha + \beta h(x)}\right)^n \int_{-\infty}^{\infty} \frac{1}{2} \mu e^{-\mu|y-x|} u(y, \alpha)\, dy, \tag{4.1}$$

where $h(x) = 0$ for $x < \ell$ and 1 for $x \geqslant \ell$.

To find u, we follow Example 4 of [1] in which Snyder used Wiener-Hopf techniques to find u for the case $n = 1$. Without loss of generality, we take $\ell = 0$ and define

$$v_+(x, \alpha) = \begin{cases} u(x, \alpha) - \dfrac{1}{\alpha + \beta} & x \geqslant 0 \\ 0 & x < 0, \end{cases} \qquad v_-(x, \alpha) = \begin{cases} 0 & x \geqslant 0 \\ u(x, \alpha) - \dfrac{1}{\alpha} & x < 0. \end{cases}$$

Equation (4.1) then becomes

$$(\alpha+\lambda)^n v_-(x,\alpha)+(\alpha+\lambda+\beta)^n v_+(x,\alpha) - \frac{\lambda^n}{2}\int_{-\infty}^{\infty} \mu e^{-\mu|x-y|}[v_+(y,\alpha)+v_-(y,\alpha)]\,dy = \frac{\lambda^n\beta\,\mathrm{sgn}(x)}{2\alpha(\alpha+\beta)}e^{-\mu|x|}. \tag{4.2}$$

Let

$$\hat{v}_+(\theta,\alpha)=\int_{-\infty}^{\infty} e^{i\theta x}v_+(x,\alpha)\,dx$$

$$\hat{v}_-(\theta,\alpha)=\int_{-\infty}^{\infty} e^{i\theta x}v_-(x,\alpha)\,dx,$$

where $i=\sqrt{-1}$. Taking transforms in Eq. (4.2), we have

$$(\alpha+\lambda)^n\hat{v}_-(\theta,\alpha)\,(\alpha+\lambda+\beta)^n\hat{v}_+(\theta,\alpha)-\lambda^n\left(\frac{\mu^2}{\theta^2+\mu^2}\right)[\hat{v}_+(\theta,\alpha)+\hat{v}_-(\theta,\alpha)] = \frac{\lambda^n\beta}{\alpha(\alpha+\beta)}\left(\frac{i\theta}{\theta^2+\mu^2}\right). \tag{4.3}$$

A trivial modification of the argument in Example 4 of [1] shows, by a Wiener-Hopf factorization, that

$$\hat{v}_+(\theta,\alpha)=\frac{i\lambda^n\beta\theta_1}{\alpha(\alpha+\beta)\,(\alpha+\beta+\lambda)^n\,(\theta_1+\theta_2)\,(\theta+i\theta_2)},$$

and

$$\hat{v}_-(\theta,\alpha)=\frac{i\lambda^n\beta\theta_2}{\alpha(\alpha+\beta)\,(\alpha+\lambda)^n\,(\theta_1+\theta_2)\,(\theta-i\theta_2)},$$

where

$$\theta_1=\mu\left[\frac{(\alpha+\lambda)^n-\lambda^n}{(\alpha+\lambda)^n}\right]^{1/2}, \qquad \text{and} \qquad \theta_2=\mu\left[\frac{(\alpha+\lambda+\beta)^n-\lambda^n}{(\alpha+\lambda+\beta)^n}\right]^{1/2}.$$

Inverting $\hat{v}_+$ and $\hat{v}_-$ we have

$$v_+(x,\alpha)=\frac{\lambda^n\beta\theta_1e^{-\theta_2x}}{(\theta_1+\theta_2)\,\alpha(\alpha+\beta)\,(\alpha+\beta+\lambda)^n} \qquad \text{for} \qquad x\geqslant 0,$$

and (4.4)

$$v_-(x,\alpha)=\frac{-\lambda^n\beta\theta_2e^{\theta_1x}}{(\theta_1+\theta_2)\,\alpha(\alpha+\beta)\,(\alpha+\lambda)^n} \qquad \text{for} \qquad x<0.$$

In the case of an arbitrary level, one need only replace x by $x - \ell$ in the above equations to obtain v_- and v_+. Thus, for an arbitrary level ℓ, we have that

$$u(x, \alpha) = \begin{cases} \dfrac{1}{\alpha + \beta}\left\{1 + \dfrac{\lambda^n \beta \theta_1}{(\theta_1 + \theta_2)\,\alpha(\alpha + \beta + \lambda)^n} \exp[-\theta_2(x - \ell)]\right\} & \text{for} \quad x \geqslant \ell \\ \dfrac{1}{\alpha}\left\{1 - \dfrac{\lambda^n \beta \theta_2}{(\theta_1 + \theta_2)\,(\alpha + \beta)\,(\alpha + \lambda)^n} \exp[\theta_1(x - \ell)]\right\} & \text{for} \quad x < \ell \end{cases}$$

is the bounded solution of (4.1).

From the definition of u, we see that as β approaches ∞ we obtain the Laplace transform of the first passage time to or above the level ℓ. Using (4.5) and (3.6), we have

$$\int_0^\infty e^{-\alpha t} M_x(t)\, dt = \lim_{\beta \to \infty} u(x, \alpha) = \begin{cases} 0 & \text{for} \quad x \geqslant \ell \\ \dfrac{1}{\alpha}\left\{1 - \left(1 - \dfrac{\theta_1}{\mu}\right) \exp[\theta_1(x - \ell)]\right\} & \text{for} \quad x < \ell. \end{cases} \tag{4.6}$$

We note that (4.6) could be obtained directly by solving Eq. (3.10) of [4] by a Wiener-Hopf factorization similar to the one used here.

Finally, from (4.6) we have

$$\int_0^\infty e^{-\alpha t} P_{\ell,0}[X_s \leqslant \ell, 0 \leqslant s \leqslant t]\, dt = \lim_{x \uparrow \ell} \lim_{\beta \to \infty} u(x, \alpha) = \frac{1}{2}\left[\frac{(\alpha + \lambda)^n - \lambda^n}{(\alpha + \lambda)^n}\right]^{1/2}. \tag{4.7}$$

In the case where $n = 1$, (4.7) may be inverted to find

$$P_{\ell,0}[X_s \leqslant \ell, 0 \leqslant s \leqslant t] = e^{-\lambda/2t} I_0\left(\frac{\lambda}{2} t\right),$$

where I_0 is the 0-th order modified Bessel function of the first kind. This was noted previously in [1]. When $n = 2$, we obtain

$$P_{\ell,0}[X_s \leqslant \ell, 0 \leqslant s \leqslant t] = e^{-\lambda t}\left[I_0(\lambda t) + \lambda \int_0^t I_0(\lambda s)\, ds\right].$$

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