INCREMENTAL APPROXIMATION OF OPTIMAL ALLOCATIONS*

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ABSTRACT

This paper concerns the approximation of optimal allocations by Δ allocations. Δ allocations are obtained by fixing an increment Δ of effort and deciding at each step upon a single cell in which to allocate the entire increment. It is shown that Δ allocations may be used as a simple method of approximating optimal allocations of effort resulting from constrained separable optimization problems involving a finite number of cells. The results are applied to find Δ allocations (called Δ plans) which approximate optimal search plans. Δ plans have the property that as $\Delta \to 0$, the mean time to find the target using a Δ plan approaches the mean time when using the optimal plan. Δ plans have the advantage that they are easily computed and more easily realized in practice than optimal plans which tend to be difficult to calculate and to call for spreading impractically small amounts of effort over large areas.

1. INTRODUCTION

Before stating our results in a mathematical fashion, we discuss the motivation for studying methods of approximating optimal allocations. The motivation arises from the study of optimal search plans.

Optimal search plans have been found for a large class of searches for a stationary object. Koopman [4] and DeGuenin [1] have found optimal plans when the search sensor has perfect discrimination. In the case of sensors with uncertain sweep width, optimal plans are given in [5]. In [7] and [8], optimal plans have been found for several classes of searches involving false targets. Typically, optimal plans have a complicated form even when one is able to overcome the analytic difficulties involved in finding their explicit expressions. This complicated form makes it difficult to provide planning advice without the aid of computers. Even in the cases where the functional form of the optimal plan is simple (see, for example, p. 41 of [3]), the plans typically call for spreading small amounts of search over ever expanding areas; however, this is very difficult to do in practice.

As a method of overcoming some of the difficulties involved in using optimal plans, we present a class of plans called Δ plans. These plans are executed in a step by step fashion. At each step one is required to calculate simple ratios and allocate Δ amount of effort to a single cell. These plans can be found by using a desk calculator and do not require that small amounts of search effort be spread over large areas. Furthermore, these plans approximate the optimal plan in the sense that as $\Delta \to 0$, the mean time to find the target using a Δ plan approaches the mean time using the optimal plan.

One major restriction on using Δ plans is that they apply only when the target location distribution is specified by a finite number of cells R_j, $1 \leq j \leq J$, such that the target is in cell R_j with probability

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p_j. For most search operations, however, the target location distribution may be specified in this manner (see [6]). In mathematical terms our results may be described as follows.

For functions, f, of two variables we shall use $f(x,\cdot)$ to indicate the function obtained from f by fixing the first variable at x. Similarly, when fixing the second variable at y, we write $f(\cdot, y)$. Let **J** be a countable set of indices which for convenience we take to be a subset of the positive integers. For $j \in \mathbf{J}$ let $e(j, \cdot)$ *and* $c(j, \cdot)$ be nonnegative functions defined on $[0,\infty)$ such that $e(j,0)=c(j,0)=0$. Let G be the set of nonnegative functions g defined on **J** and let

$$E(g)=\Sigma e(j,g(j)) \qquad \text{and} \qquad C(g)=\Sigma c(j,g(j)),$$

where summations without indicated ranges are understood to run over all of **J**. Let J denote the cardinality of **J**.

An *allocation* is a function $q\colon \mathbf{J}\times[0,\infty)\to[0,\infty)$ such that

(i) $q(j,\cdot)$ is increasing and continuous for $j\in\mathbf{J}$

(ii) $\Sigma\, q(j,s)=s \qquad$ for $s\geqslant 0$.

We refer to $q(j, s)$ as the amount of effort in the jth cell, R_j, to $c(j, q(j, s))$ as the cost of that effort and to $e(j, q(j, s))$ as the effectiveness resulting from the effort. Let Q be the set of all allocations q. Thus $C(q(\cdot,s))$ and $E(q(\cdot,s))$ are the global cost and effectiveness resulting from $q(\cdot,s)$. Define

$$\mu(q)=\int_0^\infty C(q(\cdot,s))\,dE(q(\cdot,s)),$$

whenever the integral exists in the Stieltjes sense. Then q^* is an *optimal* allocation if

$$\mu(q^*)\leqslant\mu(q) \qquad \text{for } q\in Q.$$

We shall refer to $\mu(q)$ as the *mean cost* of the allocation q.

If f is a real valued function of one or more variables, we let f' denote the partial derivative of f with respect to the last variable. In Section 2, Theorem 2.1 finds the optimal allocation q^* under the conditions that for $j\in\mathbf{J}$

(1.1)
(a) $e(j,\cdot)\leqslant B(j)$ and $\Sigma B(j)<\infty \qquad$ for some $B\colon\mathbf{J}\to[0,\infty)$.
(b) $e'(j,\cdot)$ and $c'(j,\cdot)$, are continuous and $c'(j,\cdot)>k \qquad$ for some $k>0$.
(c) $r_j=e'(j,\cdot)/c'(j,\cdot)$ is positive, continuous, and strictly decreasing.

The notion of a Δ allocation is defined in Section 3. Let μ_Δ be the mean cost resulting from a Δ allocation. Then Theorem 3.1 shows that if (1.1) is satisfied, $J<\infty$ and $c'(j,\cdot)\leqslant K$ for $j\in\mathbf{J}$, then

$$\mu_\Delta\leqslant\mu(q^*)+2(J+1)K\Delta. \tag{1.2}$$

In Section 4 we show how the results of Section 3 may be used to approximate optimal search plans by Δ allocations which are called Δ plans in this special case. Section 5 demonstrates by example that the bound in (1.2) has the correct order of magnitude in the sense that there are Δ plans such that $\mu_\Delta-\mu(q^*)$ goes linearly to infinity as $\Delta\to\infty$.

2. OPTIMAL ALLOCATION

In Theorem 2.1 of this section, we find the optimal allocation q^* under the conditions given by (1.1). Theorems 2.1 and 2.2 are generalizations of Theorem 2 of [7] to allow more general cost functions, c. Observe that if (1.1) holds, then $\mu(q)$ is well defined for all $q \epsilon Q$. In addition we may define r_j^{-1} to be the inverse function for r_j and extend its domain to $(0, \infty)$ by letting $r_j^{-1}(z) = 0$ for $z > r_j(0)$ and $r_j^{-1}(z) = \infty$ for $0 < z \leq \lim_{y \to \infty} r_j(y) = \beta_j$.

THEOREM 2.1: Assume conditions (1.1) are satisfied. Define $A(\gamma) = \sum r_j^{-1}(\gamma)$ for $\gamma > 0$. Then λ, the inverse function for A, is well defined on $(0, \infty)$. If

(2.1) $$q^*(j, s) = r_j^{-1}(\gamma(s)) \qquad \text{for } s > 0, j \epsilon \mathbf{J},$$

then $q^* \epsilon Q$,

(2.2) $$E(g) \geq E(q^*(\cdot, s)) \text{ implies } C(g) \geq C(q^*(\cdot, s)) \qquad \text{for } g \epsilon G$$

and

(2.3) $$\mu(q^*) \leq \mu(q) \qquad \text{for any } q \epsilon Q.$$

PROOF: Observe that

$$\frac{B(j)}{k} \geq \int_0^z \frac{e'(j, y) dy}{k} \geq \int_0^z \frac{e'(j, y)}{c'(j, y)} dy \geq z r_j(z) \qquad \text{for } z \geq 0.$$

Thus

$$r_j(z) \leq \frac{B(j)}{k} \frac{1}{z}, \qquad \text{and} \qquad r_j^{-1}(\gamma) \leq \frac{B(j)}{k} \frac{1}{\gamma} \qquad \text{for } \gamma > \beta_j.$$

Now we may write, for $\gamma > \max \{\beta_j : j \epsilon \mathbf{J}\}$,

(2.4) $$A(\gamma) = \sum_{[j : r_j^{-1}(\gamma) > 0]} r_j^{-1}(\gamma) \leq \frac{1}{k\gamma} \sum B(j) < \infty.$$

By using the monotone convergence theorem, one may show that $\lim_{\gamma \to \infty} A(\gamma) = 0$, $\lim_{\gamma \to 0+} A(\gamma) = \infty$ and that A is continuous. Furthermore A is strictly decreasing on the domain (Λ_l, Λ_u) where

$$\Lambda_l = \sup \{\Lambda : A(\Lambda) = \infty\} \qquad \Lambda_u = \inf \{\Lambda : A(\Lambda) = 0\}.$$

Thus there is a unique function $\lambda: (0, \infty) \to (\Lambda_l, \Lambda_u)$ such that $A(\lambda(s)) = s$ for $s > 0$. Moreover, λ is continuous and strictly decreasing.

Set $q^*(j, 0) = 0$. Then $q^*(j, \cdot)$ is continuous and increasing and $\sum q^*(j, s) = s$ for $s > 0$. Hence $q^* \epsilon Q$. One may check that for each $s > 0$, q^* satisfies

$$e'(j, z) \geq \lambda(s) c'(j, z) \qquad 0 < z < q^*(j, s)$$

$$e'(j, z) \leq \lambda(s) c'(j, z) \qquad q^*(j, s) < z < \infty, \qquad \text{for } j \epsilon \mathbf{J}.$$

Thus $q^*(\cdot, s)$ satisfies the form of the Neyman-Pearson conditions given in [9], and by Theorem 1 and Remark 4 of [9], q^* satisfies (2.2) for each $s > 0$. The proof that (2.3) holds now follows in exactly the same manner as that given in the proof of Theorem 2 of [7]. This proves the theorem.

While the form of Theorem 2.1 given above is most convenient for our use, the theorem remains true in greater generality. In particular suppose that X is a topological space and that ν is a nonnegative finite measure defined on the Borel subsets $\mathscr{S}$ of X. Let e and c be Borel functions defined on $X \times (0, \infty)$, such that $e(x, 0) = c(x, 0) = 0$ for $x \epsilon X$. Then for nonnegative Borel functions g defined on X, we let

$$E(g) = \int_X e(x, g(x)) d\nu(x) \qquad \text{and} \qquad C(f) = \int_X c(x, g(x)) d\nu(x).$$

A simple modification of the proof of Theorem 2.1 yields the following theorem.

THEOREM 2.2: Let $|e| \leq B < \infty$. For $x \epsilon X$, let $e(x, \cdot)$ and $c(x, \cdot)$ be absolutely continuous and $r(x, \cdot) = e'(x, \cdot)/c'(x, \cdot)$ be positive, continuous and strictly decreasing. Let $r^{-1}(x, \cdot)$ be the inverse function for $r(x, \cdot)$ and extend the domain of $r^{-1}(x, \cdot)$ to $(0, \infty)$ by defining $r^{-1}(x, z) = 0$ for $z > r(x, 0)$ and $r^{-1}(x, z) = \infty$ for $z < \lim_{y \to \infty} r(x, y)$. If $c'(x, \cdot) > k$ for $x \epsilon X$, and some $k > 0$, then there exists a decreasing function λ defined on $(0, \infty)$ such that

$$q^*(x,s) = r^{-1}(x, \lambda(s)) \qquad \text{for } s > 0,\ x \epsilon X$$

satisfies

$$\int_X q^*(x,s) d\nu(x) = s,$$

and for any nonnegative Borel function g,

$$E(g) \geq E(q^*(\cdot, s)) \text{ implies } C(g) \geq C(q^*(\cdot, s)).$$

3. APPROXIMATION BY Δ ALLOCATIONS

We now define Δ allocations and show that they approximate the optimal allocation in mean cost. The advantage of a Δ allocation is that it calls for fixing an increment Δ and at each step in the plan allocating Δ amount of effort to only one cell. In contrast, the optimal allocation calls for spreading smaller and smaller amounts of effort over larger and larger areas. Thus, a Δ allocation is more likely to be operationally feasible.

To retain generality, we use effort as an undefined term. Possible definitions of effort include time, man-hours, money, track length, etc. In order to perform a Δ allocation, one fixes a positive number Δ. At each step, one allocates Δ amount of effort (in some fixed units) to a single cell as follows: Calculate

$$r_j(z_j) = \frac{e'(j,z_j)}{c'(j,z_j)} \qquad \text{for } j\epsilon\mathbf{J},$$

where z_j is the amount of effort placed in R_j. Allocate the next Δ amount of effort to the cell R_j having the highest value of $r_j(z_j)$. If there is more than one cell with the highest value, then one may allocate Δ to any one of these highest cells; one possible procedure is to choose the cell with the lowest index. Any plan generated in the above manner is called a Δ allocation. The mean cost of such an allocation is denoted μ_Δ. Fix Δ and a Δ allocation. Let $h(j,n)$ be the effort placed in R_j after n steps of the Δ allocation. In order to show that (1.2) holds we prove the following lemma.

LEMMA 3.1: Suppose the conditions of (1.1) are satisfied and that $J < \infty$. Then

$$\text{(3.1)} \qquad q^*(j,(n-J)\Delta) \leq h(j,n) \leq q^*(j,n\Delta) + \Delta, \qquad \text{for } j\epsilon\mathbf{J}$$

where we define $q^*(j, (n-J)\Delta) = 0$ when $n < J$.

PROOF: Let $\lambda_n^+ = \max r_j(h(j,n))$ where max without an indicated range is understood to run over all of $\mathbf{J}$. Then we claim

$$\text{(3.2)} \qquad r_j^{-1}(\lambda_n^+) \leq h(j,n) \leq r_j^{-1}(\lambda_n^+) + \Delta, \qquad j\epsilon\mathbf{J}.$$

To see that the left-hand side of (3.2) holds we observe that $\lambda_n^+ \geq r_j(h(j,n))$ by definition and then apply r_j^{-1} to both sides of this inequality. Since r_j^{-1} is decreasing, the left-hand side of (3.2) follows.

If $h(j,n) = 0$, we observe that the righthand side of (3.2) holds trivially. Suppose $h(j,n) = i\Delta$ for some $i \geq 1$. It follows that

$$\text{(3.3)} \qquad r_j((i-1)\Delta) \geq \lambda_n^+;$$

for if (3.3) does not hold, the last increment Δ placed in R_j was not added according to a Δ allocation. That is effort was added in R_j at a time when $r_j((i-1)\Delta)$ was not the highest ratio.

Thus

$$h(j, n) = (i-1)\Delta + \Delta \leq r_j^{-1}(\lambda_n^+) + \Delta$$

and (3.2) holds.

Summing (3.2) we obtain

$$\text{(3.4)} \qquad \Sigma r_j^{-1}(\lambda_n^+) \leq n\Delta \leq \Sigma r_j^{-1}(\lambda_n^+) + J\Delta.$$

Hence $(n-J)\Delta \leq \Sigma r_j^{-1}(\lambda_n^+)$. Letting $s = (n-J)\Delta$, we have by Theorem 2.1 that

$$q^*(j, (n-J)\Delta) = r_j^{-1}(\lambda(s)) \qquad \text{and} \qquad \Sigma r_j^{-1}(\lambda(s)) = (n-J)\Delta.$$

Since $\Sigma r_j^{-1}(\lambda(s)) \leq \Sigma r_j^{-1}(\lambda_n^+)$, it follows that $\lambda_n^+ \leq \lambda(s)$ and $q^*(j, (n-J)\Delta) = r_j^{-1}(\lambda(s)) \leq r_j^{-1}(\lambda_n^+)$. Thus we have shown the left-hand inequality in (3.1). By (3.4) and a similar argument, $\Sigma r_j^{-1}(\lambda_n^+) \leq n\Delta$ and $r_j^{-1}(\lambda_n^+) \leq q^*(j, n\Delta)$. The righthand side of (3.1) now follows from (3.2), and the lemma is proved.

Since a Δ allocation specifies the allocation of effort only at integer multiples of Δ, it is convenient to choose an allocation $q_\Delta \epsilon Q$, such that

$$q_\Delta(j, n\Delta) = h(j, n), \qquad \text{for } j \epsilon \mathbf{J}, \; n = 0, 1, 2, \ldots .$$

Such an allocation q_Δ clearly exists. Let

$$E_\Delta(s) = E(q_\Delta(\cdot, s)), \; C_\Delta(s) = C(q_\Delta(\cdot, s)),$$

and

$$E^*(s) = E(q^*(\cdot, s)), \; C^*(s) = C(q^*(\cdot, s)).$$

We now find a bound on the mean cost of using a Δ allocation.

THEOREM 3.2: Suppose that the conditions of (1.1) are satisfied, that $J < \infty$ and that $c'(j, \cdot) \leqslant K$ for $j \epsilon \mathbf{J}$. Then for any Δ allocation

$$\mu_\Delta \leqslant \mu(q^*) + 2(J+1)K\Delta. \tag{3.5}$$

PROOF: From (3.1) we obtain

$$E_\Delta(n\Delta) \geqslant E^*((n-J)\Delta) \qquad \text{for } n \geqslant 1,$$

where for convenience we define $E^*(s) = E_\Delta(s) = 0$ for $s \leqslant 0$. Thus for $(n-1)\Delta \leqslant s \leqslant n\Delta$,

$$E_\Delta(s) \geqslant E^*((n-J-1)\Delta) \geqslant E^*(s-(J+1)\Delta),$$

and

$$E_\Delta(s) \geqslant E^*(s-(J+1)\Delta) \qquad \text{for } s \geqslant 0. \tag{3.6}$$

Using the other half of (3.1) we obtain

$$\begin{aligned} C_\Delta(n\Delta) &\leqslant \Sigma c(j, q^*(j, n\Delta) + \Delta) \\ &\leqslant C(q^*(\cdot, n\Delta)) + JK\Delta. \end{aligned}$$

Thus for $(n-1)\Delta \leqslant s \leqslant n\Delta$,

$$\begin{aligned} C_\Delta(s) &\leqslant C^*(n\Delta) + JK\Delta \\ &\leqslant C^*((n-1)\Delta) + (J+1)K\Delta \\ &\leqslant C^*(s) + (J+1)K\Delta, \end{aligned}$$

and

$$C_\Delta(s) \leqslant C^*(s) + (J+1)K\Delta \qquad \text{for } s \geqslant 0. \tag{3.7}$$

Since E_Δ is monotone and continuous and C_Δ is monotone, one may use integration by parts and an argument similar to the one given to prove Lemma 1 on page 148 of [2] to verify that

$$\mu_\Delta = \int_0^\infty [1 - E_\Delta(s)] dC_\Delta(s). \tag{3.8}$$

Thus by (3.8) and (3.6)

$$\mu_\Delta \leq \int_0^\infty [1-E^*(s-(J+1)\Delta)]dC_\Delta(s). \tag{3.9}$$

An argument similar to the one yielding (3.8) combined with (3.7) gives

$$\int_0^\infty [1-E^*(s-(J+1)\Delta)]dC_\Delta(s)$$

$$= \int_0^\infty C_\Delta(s)dE^*(s-(J+1)\Delta) \leq (J+1)K\Delta + \int_0^\infty C^*(s)dE^*(s-(J+1)\Delta). \tag{3.10}$$

Since

$$C^*(s) \leq C^*(s-(J+1)\Delta)+(J+1)K\Delta,$$

(3.10) yields

$$\mu_\Delta \leq \mu(q^*)+2(J+1)K\Delta$$

and the theorem is proved.

For the special case where $c(j,z)=z$, we prove the following result which is stronger than Theorem 3.2.

THEOREM 3.3: Suppose $J<\infty$ and that for $j \in \mathbf{J}$, $c(j,z)=z$ for $z \geq 0$ and $e'(j,\cdot)$ is continuous and strictly decreasing. Then

$$\mu_\Delta \leq \mu(q^*)+(J+1)\Delta. \tag{3.11}$$

PROOF: The proof proceeds in the same manner as that of Theorem 3.2 to obtain (3.6). Since $c(j,z)=z$, we have $C_\Delta(s) = C^*(s)$ and

$$\mu_\Delta = \int_0^\infty [1-E_\Delta(s)]ds \leq \int_0^\infty [1-E^*(s-(J+1)\Delta)]ds$$

which proves the Theorem.

4. APPROXIMATION OF OPTIMAL SEARCH PLANS

In this section we show how Δ allocations may be used to approximate optimal search plans in the case where the search region consists of a finite number of cells R_j, $j=1,2,\ldots,J$. The jth cell has probability p_j of containing the target and $\Sigma p_j=1$.

In order to make the search models of [7] or [8] fit into the framework developed in this paper, one must consider target location distributions which can be divided into a finite number of cells such that the density f of the distribution is constant over each cell. In many operations, the target location distribution is given in this manner (see [6]).

In order to make this paper self-contained, we briefly present the search model of [7] in a form which fits into the situation considered in this paper.

We consider search for a stationary target. The search may be complicated by the possibility of detecting false targets (i.e., objects which are not the target but cause a sensor reponse which cannot be distinguished from that of the target without further investigation). When a false target or a real target is detected, it becomes a *contact.*

The search takes place in two phases. The *broad search* phase is conducted using a sensor which can detect the target but not positively identify it. In order to investigate a contact, the broad search must stop and a *contact investigation* must begin. Once a contact investigation has begun, it must continue until the contact is identified. This is called *uninterrupted contact investigation.* At the end of a random time, the contact is correctly identified either as being or not being the target. We assume that investigation of one contact makes no contribution to investigation of any other contact or to the broad search. Also, once a contact has been investigated it is, in effect, eliminated and will not be classified as a contact if detected again.

We distinguish between two types of search time. Cumulative broad search time will be denoted by s. Cumulative time spent in all aspects of search and investigation will be denoted by t. To avoid confusion, we say a target (real or false) is *contacted* when it appears as a contact, and that it has been *identified* when contact investigation shows that contact to be a real or false target. We say the target has been *found* when it has been contacted and identified.

For the broad search it is assumed that for each cell R_j there is a *local effectiveness function* b_j defined on $[0, \infty)$ such that

(4.1)
(i) $0 \leq b_j \leq 1$, $b_j(0)=0$, and $\lim_{z\to\infty} b_j(z)=1$

(ii) b_j', the derivative of b_j, exists, is strictly positive, continuous, and strictly decreasing.

If z amount of time is spent broad searching in R_j, then $b_j(z)$ is the probability of detecting the target given it is located in R_j. The assumption that b_j depends only on the amount of search time carries with it the implicit assumption that effort is applied uniformly over R_j.

For each cell R_j, we suppose that there is a $\delta_j \geq 0$, such that

$$Pr\left\{\begin{array}{c}\text{detecting exactly } k \text{ false targets in } R_j\\ \text{in } z \text{ amount of time spent broad}\\ \text{searching in } R_j\end{array}\right\} = e^{-\delta_j b_j(z)}\frac{[\delta_j b_j(z)]^k}{k!} \qquad \text{for } k=0, 1, \ldots.$$

Moreover, we assume that the mean time required to investigate a contact found in R_j is $T_j < \infty$.

If we take $\delta_j = 0$ for $j \in \mathbf{J}$, then the above model reduces to a discrete search region version of the model given by DeGuenin [1] with the exception that DeGuenin erroneously omitted the assumption that b_j' be continuous.

A *search plan* is an allocation of a broad search time and a method of identifying contacts. For searches involving uninterrupted contact investigation we shall always assume that contact investigation is immediate. Thus the allocation of broad search completely specifies a search plan when using immediate and uninterrupted contact investigation. One may show by the same argument as that given in Section 3 of [7], that immediate contact investigation coupled with the q^* resulting from the definitions of e and c given in (4.2) below produces the smallest mean time to find the target when contact investigation is uninterrupted.

Define

$$e(j,z) = p_j b_j(z)$$

(4.2)

$$c(j,z) = z + T_j \delta_j b_j(z) \qquad j \epsilon \mathbf{J}, \qquad z \geqslant 0.$$

Then $E(q(\cdot, s))$ is the probability of detecting the target by broad search time s using plan q and $C(q(\cdot, s))$ is the expected amount of time spent in broad search and contact investigation by broad search time s given the target has not been detected. Finally $\mu(q)$ is the mean time to detect the target using plan q. Since the contact which is the target is investigated immediately in all plans, minimizing μ is equivalent to minimizing the mean time to find the target.

Since e and c defined in (4.2) satisfy the conditions of Theorem 2.1, q^* is given by (2.1) where r_j^{-1} is the suitably extended inverse of

$$r_j(z) = \frac{p_j b_j'(z)}{1 + T_j \delta_j b_j'(z)} \qquad \text{for } z > 0.$$

Define a Δ plan as follows:

Allocate each increment of Δ units of broad search time to the cell having the highest value of $r_j(z_j)$, *where* z_j *is the amount of broad search time previously placed in* R_j *and investigate the resulting contacts until they are identified.*

COROLLARY 4.1: If $J < \infty$ and

$$\max \{1 + T_j \delta_j b_j'(0)\} = K < \infty$$

then

(4.3)

$$\mu_\Delta \leqslant \mu^* + 2(J+1)K\Delta.$$

PROOF: The corollary follows directly from Theorem 3.2.

A similar although more complicated method may be used to approximate the optimal search plans generated in Section 4.1 of [8]. These Δ plans would call for adding Δ amount of broad search to the cell R_j having the highest value of

$$r_j(z_j, w_j) = \frac{p_j b_j'(z_j) a_j(w_j)}{1 + \delta_j b_j'(z_j) \alpha_j(w_j)},$$

where a_j and α_j are defined as the obvious analogs of a and α in [8], z_j gives the amount of time spent broad searching in R_j and w_j gives the amount of investigation time one is willing to devote to each contact generated in R_j. Suppose that j_o is the index of the cell to which the search is added. One then finds w_o such that

$$r_{j_o}(z_{j_o} + \Delta, w_o) = \frac{p_j a_j'(w_o)}{\delta[1 - A_j(w_o)]},$$

where A_j is defined as the analog of A in [8]. One is then willing to devote up to w_o total amount of contact investigation effort to each contact in R_j. Thus w_j becomes a function of z_j. We denote this by writing $w_j = v_j(z_j)$. If

$$\infty > K > 1 + \delta_j b_j'(z_j)[1 - A_j(v_j(z))] v_j'(z) \qquad \text{for } j \epsilon \mathbf{J} \text{ and } z \geqslant 0,$$

then (4.3) holds.

5. EXAMPLE

When there are no false targets (i.e., $\delta_j=0$ for $j\in\mathbf{J}$) Theorem 3.3 says that the penalty, in mean time, resulting from using a Δ plan is bounded by $(J+1)\Delta$. Obviously, as J or Δ approach infinity, this bound also approaches infinity. Since the bounds in the above analysis are somewhat rough, one might wonder if it is really possible for Δ plans to incur penalties which are linear in J and Δ as these quantities approach infinity. The example below answers this question in the affirmative by showing a case in which the penalty approaches infinity at the same rate as $(J\Delta)/2$.

Consider the situation where the target location distribution is uniform over a square of unit area. We suppose that this square is subdivided into J rectangles, each having equal area. We enumerate these rectangles in some order and let R_j be the jth rectangle. Thus, $p_j=1/J$, $1\leqslant j\leqslant J$. For $1\leqslant j\leqslant J$, let

$$b_j(z)=1-e^{-Jz} \qquad \text{for } z\geqslant 0$$
$$\delta_j=0.$$

(Note that all times are assumed to be given in terms of a fixed time unit.) One may check that $\mu^*=1$ for this example.

We now consider a Δ plan which allocates the $(nJ+j)$th increment of broad search time to R_j for $n\geqslant 0$. Let μ_j be the mean time to find the target given the target is located in R_j. Consider the 1st rectangle. The mean time required to find the target given it is found during the first increment Δ of effort is

$$\frac{1-(1+J\Delta)e^{-J\Delta}}{J(1-e^{-J\Delta})}.$$

Thus

$$\mu_1=\frac{1-(1+J\Delta)e^{-J\Delta}}{J}+e^{-J\Delta}(J\Delta+\mu_1).$$

Solving for μ_1, we find

$$\mu_1=\frac{1}{J}+\frac{(J-1)\Delta e^{-J\Delta}}{(1-e^{-J\Delta})}.$$

It is easily seen that

$$\mu_j=(j-1)\Delta+\mu_1,\ j=1,\ \ldots,J,$$

and

$$\mu_\Delta=\frac{1}{J}\sum_{j=1}^{J}\mu_j=\mu_1+\frac{(J-1)\Delta}{2}=\frac{1}{J}+\frac{(J-1)\Delta(1+e^{-J\Delta})}{2(1-e^{-J\Delta})}.$$

Since $\mu^*=1$,

$$\mu_\Delta=\mu^*+\frac{(J-1)\Delta}{2}\left[\frac{(1+e^{-J\Delta})}{(1-e^{-J\Delta})}-\frac{2}{J\Delta}\right].$$

Thus, the penalty resulting from using a Δ plan approaches infinity at the same rate as $J\Delta/2$ as J or Δ approaches infinity.

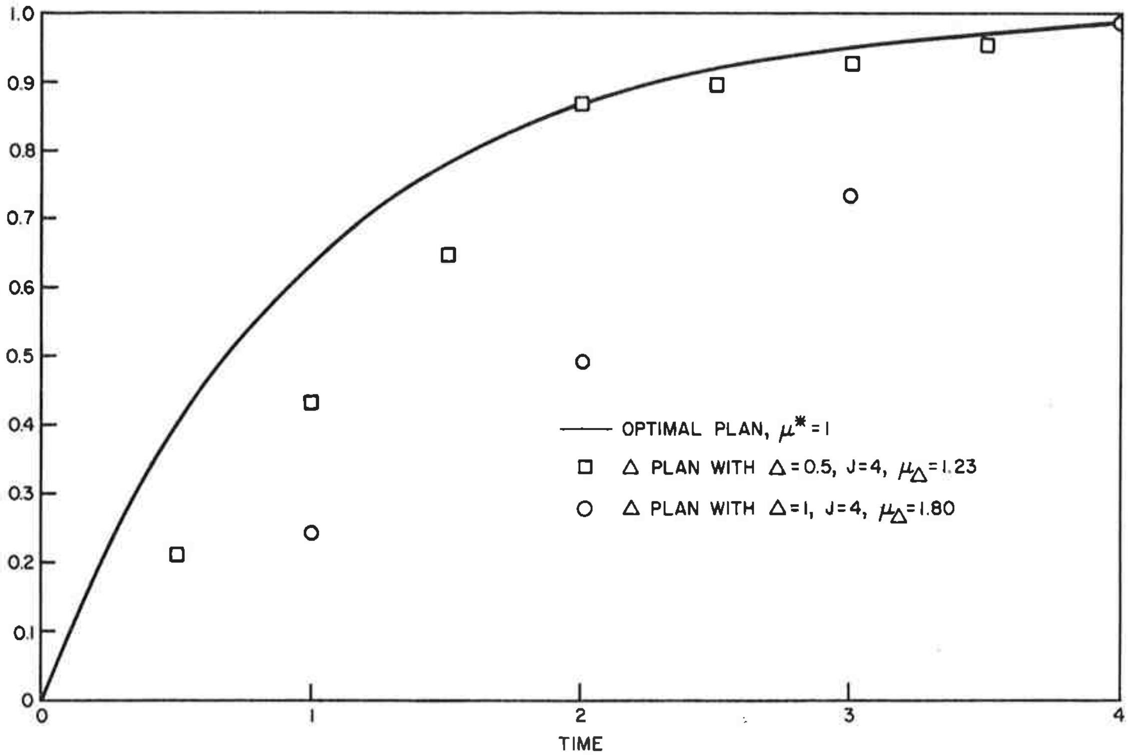

FIGURE 1. Probability of detection

In Figure 1, we have plotted the detection probabilities resulting from two of the Δ plans described above in the case where the search region is divided into four subregions. These probabilities are compared with the probabilities resulting from the optimal plan. For the first Δ plan the increment $\Delta=0.5$ has been chosen. Thus, the first 0.5 units of time are spent searching in R_1. The resulting probability of detection is 0.22. Note that the probability of detection for this plan and the optimal plan coincide at times 2 and 4. In most searches, however, the detection probability resulting from a Δ plan will always be strictly less than optimal. The second Δ plan results from taking $\Delta=1$, and as one would expect, it produces uniformly lower detection probabilities than either the optimal plan or the Δ plan with $\Delta=0.5$.

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