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INCREMENTAL AND TOTAL OPTIMIZATION OF SEPARABLE FUNCTIONALS WITH CONSTRAINTS*

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Abstract. Functionals E (real-valued) and C (vector-valued) are defined by $E(q) = \int_X e(x, q(x))\mu(dx)$ and $C(q) = \int_X c(x, q(x))\mu(dx)$, where μ is a Borel regular, nonatomic measure defined on a Borel subset X of a complete separable metric space. Let ω be the positive integers. Let $q_0, q_1, \cdots$, be extended real functions such that $q_0 = -\infty$ and $q_i \geqq q_{i-1}$ for $i \in \omega$. A function q^* is called *optimal* if $E(q^*) = \max\{E(q): C(q) = C(q^*)\}$. The sequence $(q_1, q_2, \cdots)$ is *incrementally optimal* if $E(q_i) = \max\{E(p): p \geqq q_{i-1}$ and $C(p) = C(q_i)\}$ for $i \in \omega$ and *totally optimal* if q_i is optimal for $i \in \omega$. Under appropriate measurability assumptions, it is shown that if $c(x, \cdot)$ is real-valued and increasing for $x \in X$, then an incrementally optimal sequence such that $|E(q_i)| < \infty$ and $C(q_i) \in$ interior range C for $i \in \omega$ is totally optimal. A counterexample is given to show that an extension of this result to multiple constraints fails even if $e(x, \cdot)$ and $c(x, \cdot)$ are linear for $x \in X$. In the case of a single constraint, the existence of optimal functions is proved under conditions which allow the range of C to be unbounded above.

1. Introduction. This paper investigates the relationship between incremental and total optimality for constrained optimization problems involving a real-valued separable effectiveness functional and a vector-valued separable cost functional. Existence of optimal functions is also considered.

A primary motivation for this investigation arises from search theory. In mathematical terms, the problem of finding the optimum distribution of search effort to detect a stationary object located in a subset X of Euclidean n-space becomes: find a function $q^*: X \to [0, \infty)$ such that $\int_X c(x, q^*(x))\, dx \leqq \Phi$ and

$$\int_X b(x, q^*(x)) f(x)\, dx = \max\left\{\int_X b(x, q(x)) f(x)\, dx : q \geqq 0 \quad \text{and} \quad \int_X c(x, q(x)) \leqq \Phi\right\}. \tag{1.1}$$

In (1.1), the function f gives the probability density of the target's location, $b(x, \cdot)$ is the local effectiveness function and $c(x, \cdot)$ the cost density function. In probability terms, $b(x, y)$ gives the conditional probability of detecting the target given it is located at x and the effort density is y at x. The above problem has an obvious analogue in case the search space X is discrete.

For the case where $b(x, y) = 1 - e^{-y}$ and $c(x, y) = y$ for $x \in X$ and $y \geqq 0$, Koopman [6, p. 617] made the following observation. Suppose one allocates Φ_1 amount of effort in an optimal fashion but fails to detect the target. An increment Φ_2 of effort then becomes available. If one allocates this additional effort in an incrementally optimal manner (i.e., optimal considering the previous allocation of Φ_1 amount of effort), then one obtains an optimal allocation of $\Phi_1 + \Phi_2$ effort. That is, two incrementally optimal allocations produce a totally optimal allocation. Koopman commented "This very convenient state of affairs seems to be a

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characteristic property of the basic exponential law of search [i.e., local effectiveness function] assumed throughout."

For the case where $c(x, y) = y$ for $y \geqq 0$ and $x \in X$, the following results are known. Theorem 2.1 of [7] shows that for virtually any local effectiveness function, incrementally optimal allocations are totally optimal whenever the target's probability distribution is given by a density function as in (1.1). In the case where X is discrete, it is proven that concavity of $b(x, \cdot)$ for $x \in X$ guarantees that incrementally optimal allocations are totally optimal, and it is shown by counterexample that this property need not hold for discrete X if $b(x, \cdot)$ is not concave. Another paper on this subject is [3] (see discussion in [7]).

The results in [7] are very satisfactory for the case where $c(x, y) = y$ for $y \geqq 0$ and $x \in X$. However, search problems are not limited to this situation. Thus one is led to ask if these results hold for most general cost functions. In this paper, two possible generalizations are considered. The first, Theorem 3.1, generalizes the result of Theorem 2.1 of [7] to allow $c(x, \cdot)$ to be real-valued and increasing. In Example 3.2 it is shown by counterexample that the assumption that $c(x, \cdot)$ is increasing cannot be dropped. The second possible generalization is to allow for multiple constraints. That is, we allow $c(x, \cdot)$ to be vector-valued. For example, one might consider a search in which there is a constraint on both cost (in dollars) and time. For vector-valued cost functions, it is shown in Example 3.3 that no extension of the results of Theorem 3.1 is possible even if one assumes the linearity of $c(x, \cdot)$ and $e(x, \cdot)$ for $x \in X$. Example 3.3 also shows that total optimality of incrementally optimal allocations is not a consequence of the convexity of the range of the functionals or of the satisfaction of the pointwise multiplier rule by an optimal allocation.

While motivated by search theory, the results of this paper are stated in terms of a real-valued "effectiveness" functional E subject to an equality or inequality constraint on a vector-valued "cost" functional C. These functionals are separable, which means they are given by $E(q) = \int_X e(x, q(x))\mu(dx)$ and $C(q) = \int_X c(x, q(x))\mu(dx)$, where $q(x) \in Y(x)$ for $x \in X$ and X, Y, μ, e, and c are fixed. If we take μ to be Lebesgue measure on n-space and for $x \in X$ let $Y(x) = [0, \infty)$, and $e(x, y) = f(x)b(x, y)$ for $y \in Y(x)$, then we obtain the search situation considered above.

Section 4 of this paper concerns the existence of uniformly optimal search plans. In search theory, a uniformly optimal plan is an allocation in time and space which maximizes the probability of detection at each time t. This is the most desirable search plan since one can proceed with a plan which yields the long term goal of maximizing probability of detection by the end of the time allotted for the search without sacrificing any short term gain. In Theorem 4.3, it is shown that if $b(x, \cdot)$ is increasing and right-continuous, $c(x, \cdot)$ is extended real-valued and continuous and the probability distribution of the target is given by a probability density function with respect to a nonatomic measure, then a uniformly optimal plan exists under very general conditions which cover most situations likely to occur in search theory. In fact, the uniformly optimal plan is, in a sense, constructed in Theorem 4.3.

Theorem 4.3 is stated in terms of general separable functionals (one-dimensional) E and C. One feature of interest beyond search theory in Theorem 4.3, is that it gives an existence theorem for optimal allocations without assuming that the range of C is bounded above. Theorem 4.3 is an extension of Theorem 3.3 of [7] to allow more general cost density functions c.

2. Preliminary definitions and assumptions. Throughout this paper, we assume that X is a Borel subset of a complete separable metric space and μ is a measure on X. For $x \in X$, we let $Y(x)$ be a subset of the extended real line $\bar{\mathscr{E}}_1$ with the usual topology. In order to use the results of [4] and [8], we use the definitions of measure, measurable set, and measurable function given in [4]. In order to use Corollary 5.2 of [8], we further assume that for each measurable $P \subset X$ for which $\mu(P) > 0$, there exists a measurable $Q \subset P$ such that $0 < \mu(Q) < \infty$. Following [4], we say that μ is Borel regular, if and only if all open sets of X are measurable and each set $A \subset X$ is contained in a Borel set B for which $\mu(A) = \mu(B)$.

For definiteness, the reader may wish to think of X as a Borel subset of Euclidean n-space and μ as Lebesgue measure on X. This identification will satisfy the measurability and topological hypotheses of all of the theorems in this paper. The phrase $x \in X$ is understood to mean almost every (in μ measure) $x \in X$, and a.e. stands for almost everywhere in μ measure.

Let $\Omega = \{(x, y) : x \in X \text{ and } y \in Y(x)\}$ be a Borel subset of $X \times \bar{\mathscr{E}}_1$, and let $c_1, \cdots, c_k$ and e be extended real-valued Borel functions defined on Ω. Using a framework very similar to [8] (which, however, does not permit e or c_i to assume $\pm\infty$), we let

$$\Psi = \{q : q \text{ is a function on } X \text{ and } q(x) \in Y(x) \text{ for } x \in X\},$$

$$\Xi = \{q : q \in \Psi \text{ and } c_1(\cdot, q(\cdot)), \cdots, c_k(\cdot\ q(\cdot)), e(\cdot, q(\cdot)) \text{ are measurable}\},$$

$$\Phi = \{q : q \in \Xi \text{ and } c_1(\cdot, q(\cdot)), \cdots, c_k(\cdot, q(\cdot)), e(\cdot, q(\cdot)) \text{ are integrable}\},$$

and

$$C_i(q) = \int_X c_i(x, q(x))\, \mu(dx) \quad \text{for } i = 1, \cdots, k, \quad q \in \Phi,$$

$$E(q) = \int_X e(x, q(x))\, \mu(dx) \quad \text{for } q \in \Phi.$$

Let ω be the set of positive integers. For $n \in \omega$, we let $\mathscr{E}_n$ be Euclidean n-space. If $a, b \in \mathscr{E}_n$, then $a \geqq b$ means $a_i \geqq b_i$ for $i = 1, \cdots, n$. Let $\mathscr{E}_n^+ = \{a : a \in \mathscr{E}_n$ and $a_i \geqq 0$ for $i = 1, \cdots, n\}$. If a and $b \in \mathscr{E}_n$, we denote their inner product by $a \cdot b$; this is extended to vectors with $\pm\infty$ components in the obvious way, being undefined if $0 \cdot \infty$ or if $\infty - \infty$ occurs. We let $c = (c_1, \cdots, c_k)$, $C = (C_1, \cdots, C_k)$.

We define $q^* \in \Phi$ to be *optimal* if

$$E(q^*) = \max\{E(q) : C(q) = C(q^*)\}, \tag{2.1}$$

and we say that q^* is *strongly optimal* if

$$E(q^*) = \max\{E(q) : C(q) \leqq C(q^*)\}. \tag{2.2}$$

In this and similar usage, it is understood that $E(p) \in \{E(q) : C(q) = C(q^*)\}$ implies $E(q)$ exists.

An extended real-valued function f defined on a subset of the extended real line is said to be increasing if $y \geqq x$ implies $f(y) \geqq f(x)$.

3. Incremental optimization. For $i \in \omega$, let $q_i \in \Phi$ be such that $q_1 \leqq q_2 \cdots$. Let $q_0(x) = -\infty$ for $x \in X$, and define $C(q_0) = (-\infty, \cdots, -\infty)$, whether or not $q_0 \in \Phi$. If for $i \in \omega$,

$$E(q_i) = \max \{E(q) : q \geqq q_{i-1}, q \in \Phi \text{ and } C(q) = C(q_i)\}, \tag{3.1}$$

then we say that $(q_1, q_2, \cdots)$ is an *incrementally optimal* sequence. If for $i \in \omega$,

$$E(q_i) = \max \{E(q) : q \geqq q_{i-1}, q \in \Phi \text{ and } C(q_{i-1}) \leqq C(q) \leqq C(q_i)\}, \tag{3.2}$$

then we say that $(q_1, q_2, \cdots)$ is a *strong incrementally optimal* sequence. If q_i is optimal (strongly optimal) for each $i \in \omega$, then $(q_1, q_2, \cdots)$ is said to be a *totally optimal* (*strong totally optimal*) sequence.

Conceptually, an incrementally optimal sequence $(q_1, q_2, \cdots)$ is one such that for $i \in \omega$, q_{i+1} obtains the maximum effectiveness from the increment of cost $C(q_{i+1}) - C(q_i)$ given the previous allocation q_i. If for each i, q_i is an optimal allocation of $C(q_i)$, then the sequence is totally optimal.

Under the primary conditions of a single cost constraint and increasing cost function, we show that an incrementally optimal sequence is totally optimal. Define

$$\begin{aligned} l(x, y, \lambda) &= e(x, y) - \lambda \cdot c(x, y) && \text{for } (x, y) \in \Omega \text{ and } \lambda \in \mathscr{E}_k, \\ M(x, \lambda) &= \sup \{l(x, y, \lambda) : y \in Y(x)\} && \text{for } x \in X, \quad \lambda \in \mathscr{E}_k. \end{aligned} \tag{3.3}$$

when neither $\infty - \infty$ nor $0 \cdot \infty$ occurs.

Following [8], we say that $q \in \Psi$ *satisfies* (*strongly satisfies*) *the pointwise multiplier rule* if for some $\lambda \in \mathscr{E}_k$ (some $\lambda \in \mathscr{E}_k^+$),

$$l(x, q(x), \lambda) = M(x, \lambda) \quad \text{for } x \in X.$$

In order to make use of Corollary 5.2 of [8], we note that the extended real line is a complete separable metric space under the following metric:

$$d(x, y) = |\arctan(x) - \arctan(y)| \quad \text{for } x, y \in \bar{\mathscr{E}}_1,$$

where $\arctan(-\infty) = -\pi/2$ and $\arctan(\infty) = \pi/2$.

THEOREM 3.1. *Assume Ω, e, and c are Borel. Let μ be Borel regular and non-atomic. Let e be real-valued and for $x \in X$, let $c(x, \cdot)$ be real-valued and increasing. If $(q_1, q_2, \cdots)$ is a (strong) incrementally optimal sequence such that for $i \in \omega$, $|E(q_i)| < \infty$ and $C(q_i)$ is in the interior of the range of C, then $(q_1, q_2, \cdots)$ is a (strong) totally optimal sequence.*

Proof. By Corollary 5.2 of [8], there exists a $\lambda^1 \in \mathscr{E}_1$ such that for $x \in X$,

$$l(x, q_1(x), \lambda^1) = M(x, \lambda^1) \tag{3.4}$$

and $\lambda^2 \in \mathscr{E}_1$ such that for $x \in X$,

$$l(x, q_2(x), \lambda^2) \geqq l(x, y, \lambda^2) \quad \text{for } q_1(x) \leqq y \in Y(x). \tag{3.5}$$

In order to prove that q_2 is optimal (strongly optimal), it is sufficient, by Theorem 2.1 of [8], to show that there exists $\lambda \in \mathscr{E}_1 (\lambda \geqq 0)$ such that

$$l(x, q_2(x), \lambda) = M(x, \lambda) \quad \text{for } x \in X. \tag{3.6}$$

By (3.4) and (3.5),

$$\begin{aligned} \lambda^2[c(x, q_2(x)) - c(x, q_1(x))] &\leqq e(x, q_2(x)) - e(x, q_1(x)) \\ &\leqq \lambda^1[c(x, q_2(x)) - c(x, q_1(x))] \quad \text{for } x \in X. \end{aligned} \tag{3.7}$$

If $c(x, q_1(x)) = c(x, q_2(x))$ for $x \in X$, then by (3.7), $e(x, q_1(x)) = e(x, q_2(x))$, and

$$e(x, q_2(x)) - \lambda^1 c(x, q_2(x)) = e(x, q_1(x)) - \lambda^1 c(x, q_1(x)) = M(x, \lambda^1) \quad \text{for } x \in X.$$

Thus (3.6) is satisfied for $\lambda = \lambda^1$. If $c(x, q_2(x)) > c(x, q_1(x))$ for x in a set of positive measure, then $\lambda^2 \leqq \lambda^1$ by (3.7). Suppose $y \in Y(x)$ and $y < q_1(x)$; then for $x \in X$,

$$\begin{aligned} 0 &\leqq e(x, q_1(x)) - e(x, y) - \lambda^1[c(x, q_1(x)) - c(x, y)] \\ &\leqq e(x, q_1(x)) - e(x, y) - \lambda^2[c(x, q_1(x)) - c(x, y)] \end{aligned}$$

and

$$l(x, y, \lambda^2) \leqq l(x, q_1(x), \lambda^2) \leqq l(x, q_2(x), \lambda^2) \quad \text{for } q_1(x) > y \in Y(x). \tag{3.8}$$

Combining (3.8) and (3.5), we obtain (3.6) with $\lambda = \lambda^2$. Thus q_2 satisfies the pointwise multiplier rule and, by Theorem 2.1 of [8], q_2 is optimal. By repeating the above argument for $q_3, q_4, \cdots$, the theorem is proved for optimality. The assertions concerning strong optimality follow by observing that in this case, Corollary 5.2 of [8] yields $\lambda^1, \lambda^2 \geqq 0$. Thus the number λ obtained to satisfy (3.6) is nonnegative.

Example 3.2. Theorem 3.1 does not remain true if one drops the assumption that $c(x, \cdot)$ is increasing for $x \in X$. To see this, we consider the situation where $X = [0, 1]$ and for $x \in X$, $Y(x) = [0, 3]$,

$$e(x, y) = \begin{cases} y, & 0 \leqq y \leqq 1, \\ -y + 2, & 1 \leqq y \leqq 3, \end{cases} \qquad c(x, y) = \begin{cases} y, & y \neq \frac{1}{2}, \\ 2, & y = \frac{1}{2}. \end{cases}$$

For $x \in X$, let $q_1(x) = 1$ and $q_2(x) = 2$. One may check that (q_1, q_2) is an incrementally optimal sequence by noting that for $x \in X$,

$$l(x, 1, 1) = M(x, 1),$$

$$e(x, 2) + c(x, 2) \geqq e(x, y) + c(x, y) \quad \text{for } y \geqq 1.$$

However, by taking $h(x) = \frac{1}{2}$, for $x \in X$, we find that

$$E(h) = \tfrac{1}{2} > 0 = E(q_2) \quad \text{and} \quad C(h) = 2 = C(q_2),$$

so that q_2 is not totally optimal.

A similar example can be constructed even if one requires that $c(x, \cdot)$ be continuous for $x \in X$.

Example 3.3. We now show that Theorem 3.1 cannot be extended to multiple constraints even when one requires that $e(x, \cdot)$, $c_i(x, \cdot)$ be linear for $i = 1, \cdots, k$ and $x \in X$. Since a linear function is both concave and convex, this shows that no combination of convexity/concavity assumptions will guarantee that incrementally optimal sequences are totally optimal. That strict concavity or convexity may be sufficient is not ruled out by the counterexample; however, it seems unlikely that these assumptions would suffice.

Let $X = [0, 4]$, $Y(x) = [0, \infty)$ for $x \in X$. For $y \geqq 0$, let

$$
\begin{array}{llll}
e(x, y) = y, & c_1(x, y) = 4y, & c_2(x, y) = 2y & \text{for } 0 \leqq x \leqq 1, \\
e(x, y) = \frac{3}{4}y, & c_1(x, y) = y, & c_2(x, y) = 2y & \text{for } 1 \leqq x \leqq 2, \\
e(x, y) = \frac{1}{2}y, & c_1(x, y) = y, & c_2(x, y) = \frac{9}{8}y & \text{for } 2 < x \leqq 3. \\
e(x, y) = 0, & c_1(x, y) = y, & c_2(x, y) = 4y & \text{for } 3 < x \leqq 4.
\end{array}
$$

By Blackwell's extension (see [2]) of Lyapunov's theorem, the range of C is convex. Let (z_1, z_2) represent a point in 2-space. Then one can check that the range of C is the region in the nonnegative quadrant of 2-space which lies between the line $z_2 = \frac{1}{2}z_1$ and the line $z_2 = 4y_1$. Let

$$
q_1(x) = \begin{cases} 0, & 0 \leqq x \leqq 1, \\ 1, & 1 < x \leqq 2, \\ 0, & 2 < x \leqq 4, \end{cases} \qquad q_2(x) = \begin{cases} 1, & 0 \leqq x \leqq 1, \\ 1, & 1 < x \leqq 2, \\ 0, & 2 < x \leqq 4. \end{cases}
$$

Note that $C[q_1] = (1, 2)$ and $C[q_2] = (5, 4)$ are in the interior of the range of C. By choosing $\lambda^1 = (\frac{1}{4}, \frac{1}{4})$ and $\lambda^2 = (\frac{1}{40}, \frac{9}{20})$, one may check that for $x \in X$,

$$
e(x, q_1(x)) - \lambda^1 \cdot c(x, q_1(x)) = M(x, \lambda^1)
$$

and

$$
e(x, q_2(x)) - \lambda^2 \cdot c(x, q_2(x)) \geqq e(x, y) - \lambda^2 \cdot c(x, y) \quad \text{for } y \geqq q_1(x).
$$

Thus (q_1, q_2) is a strong incrementally optimal sequence. However, q_2 is not optimal, much less strongly optimal. To see this, define

$$
h(x) = \begin{cases} 0, & 0 \leqq x \leqq \frac{7}{48}, \\ 1, & \frac{7}{48} < x \leqq 1, \\ 0, & 1 < x \leqq 1 + \frac{5}{12}, \\ 1, & 1 + \frac{5}{12} < x \leqq 3, \\ 0, & 3 < x \leqq 4. \end{cases}
$$

Then one may check that $C(h) = C(q_2) = (5, 4)$, but $E(h) = 1 + \frac{19}{24} > 1 + \frac{3}{4} = E(q_2)$.

Observe that this example satisfies the conditions of Corollary 5.2 of [8] so that optimal allocations satisfy a pointwise multiplier rule. Although this is the main tool used to prove Theorem 3.1, it is not sufficient for the analog of Theorem 3.1 to hold for multiple constraints.

4. Existence of optimal allocations. In this section, we prove the existence of uniformly optimal allocations for a single constraint involving a continuous, increasing cost function under assumptions which are natural, for example, for search theory. Lemmas 4.1 and 4.2 below will be used to prove the main existence result, Theorem 4.3. Throughout this section, we take $c(x, \cdot)$ to be extended real-valued (i.e., $k = 1$) and denote c_1 by c.

The following lemma is an extension of Halkin's Proposition 8.3 in [5] which is proved for totally finite measure spaces. The extension to σ-finite measure spaces stated in the lemma is routine.

LEMMA 4.1. *Let μ be a nonatomic σ-finite measure on X. Then there is a family $\{S_\alpha : \alpha \in [0, 1]\}$ of measurable sets such that*

(4.1) (i) $\quad S_0 = \phi, \quad S_1 = X \quad$ *and* $\quad \alpha < \beta$ *implies* $S_\alpha \subset S_\beta$,

(4.1) (ii) $\quad \mu(S_\alpha) < \infty \quad$ *for* $0 \leqq \alpha < 1$,

(4.1)(iii) $\quad \lim_{\alpha \to \beta} \mu(S_\alpha) = \mu(S_\beta) \quad$ *for* $\beta \in [0, 1]$,

(4.1)(iv) $\quad \lim_{\alpha \uparrow \beta} S_\alpha = S_\beta \quad$ *for* $\beta \in [0, 1]$.

For $x \in X$, let

$$T(x) = \inf \{y : y \in Y(x)\},$$
$$U(x) = \sup \{y : y \in Y(x)\}. \tag{4.2}$$

Suppose $Y(x)$ is compact in $\bar{\mathscr{E}}_1$, $e(x, \cdot)$ is real-valued, increasing and right-continuous and $c(x, \cdot)$ is continuous and increasing for $x \in X$. Then for $x \in X$ we define

$$\varphi(x, \lambda) = \sup \{y : y \in Y(x) \text{ and } l(x, y, \lambda) = M(x, \lambda)\} \quad \text{for } \lambda > 0,$$
$$\xi(x, \lambda) = \lim_{\lambda' \downarrow \lambda} \varphi(x, \lambda') \quad \text{for } \lambda \geqq 0.$$

Note that $Y(x)$ need not be an interval. Since $Y(x)$ is compact and $l(x, \cdot, \lambda)$ is upper semicontinuous for $\lambda > 0$, $M(x, \lambda)$ is achieved on $Y(x)$ and $\varphi(x, \lambda)$ is well-defined. We let

$$I(\lambda) = \int_X c(x, \varphi(x, \lambda))\mu(dx) \quad \text{for } \lambda > 0,$$

when the integral exists.

LEMMA 4.2. *Assume Ω, e, and c are Borel. Let μ be σ-finite, nonatomic and Borel regular. Suppose $-\infty < E(T) \leqq E(U) < \infty$, $|C(T)| < \infty$, and for $x \in X$, $Y(x)$ is compact in $\bar{\mathscr{E}}_1$, $e(x, \cdot)$ is increasing and right-continuous, and $c(x, \cdot)$ is continuous, increasing, and extended real-valued. Then the following hold:*

(a) *$\varphi(\cdot, \lambda) \in \Phi$ and $I(\lambda)$ is finite for $\lambda > 0$;*
(b) *$\varphi(x, \cdot)$ is decreasing for $x \in X$ and I is decreasing,*
(c) *$\varphi(x, \cdot)$ is left-continuous for $x \in X$ and I is left-continuous,*
(d) *For $\lambda > 0$ we may find $f: X \times [C(\xi(\cdot, \lambda)), I(\lambda)] \to \bar{\mathscr{E}}_1$ such that* (i) *$f(x, \cdot)$ is increasing for $x \in X$,* (ii) *for $t \in [C(\xi(\cdot, \lambda)), I(\lambda)]$, $C(f(\cdot, t)) = t$, and* (iii),

$$l(x, f(x, t), \lambda) = M(x, \lambda) \quad \text{for } x \in X.$$

Proof. Since $Y(x)$ is compact for $x \in X$, $U \in \Psi$. To see that $U \in \Xi$, let $\pi(x, y) = x$ for $(x, y) \in \Omega$ and $R = \Omega \cap \{(x, y) : y \geqq a\}$. Then R is Borel, and $\{x : U(x) \geqq a\} = \pi(R)$. Since X and $\bar{\mathscr{E}}_1$ are complete separable metric spaces, so is $X \times \bar{\mathscr{E}}_1$. Thus by 2.2.13 of [4], $\pi(R)$ is measurable and $U \in \Xi$. A similar argument shows $T \in \Xi$.

Since $-\infty < E(T) \leqq E(U) < \infty$, we must have $e(x, \cdot)$ real-valued for $x \in X$. The compactness of $Y(x)$ and the upper semi-continuity of $l(x, \cdot, \lambda)$ guarantee that $\varphi(x, \lambda) \in Y(x)$ and

$$l(x, \varphi(x, \lambda), \lambda) = M(x, \lambda) \quad \text{for } x \in X \quad \text{and} \quad \lambda > 0.$$

To show that $\varphi(\cdot, \lambda) \in \Xi$, take $\lambda > 0$, $a \in \bar{\mathscr{E}}_1$, and let

$$R = \Omega \cap \{(x, y) : l(x, y, \lambda) > a\}.$$

Since Ω, e, and c are Borel, R is Borel. By the same reasoning as above, $\{x : M(x, \lambda) > a\} = \pi(R)$ is measurable, and by 2.3.6 of [4], $M(\cdot, \lambda)$ is equal a.e. to a Borel function $\tilde{M}(\cdot, \lambda)$. Similarly,

$$\{x : \varphi(x, \lambda) > a\} = \pi\{(x, y) : l(x, y, \lambda) = \tilde{M}(x, y) \text{ and } y > a\}$$

is measurable and $\varphi(\cdot, \lambda) \in \Xi$.

By virtue of $|C(T)| < \infty$, we have $c(x, T(x))$ is finite for $x \in X$. Since $e(x, \cdot)$ is increasing, we have

$$\begin{aligned} e(x, U(x)) - e(x, T(x)) &\geqq e(x, \varphi(x, \lambda)) - e(x, T(x)) \\ &\geqq \lambda[c(x, \varphi(x, \lambda)) - c(x, T(x))]. \end{aligned}$$

Hence

$$-\infty < C(T) \leqq I(\lambda) \leqq (1/\lambda)[E(U) - E(T)] + C(T) < \infty,$$

which shows that I is finite, and $\varphi(\cdot, \lambda) \in \Phi$. This proves (a).

Suppose $0 < \lambda^1 < \lambda^2$. Let $y_1(x) = \varphi(x, \lambda^1)$ and $y_2(x) = \varphi(x, \lambda^2)$ for $x \in X$. Then for $x \in X$,

$$l(x, y_1(x), \lambda^1) \geqq l(x, y_2(x), \lambda^1) \quad \text{and} \quad l(x, y_2(x), \lambda^2) \geqq l(x, y_1(x), \lambda^2)$$

which implies

$$\begin{aligned} \lambda^1[c(x, y_1(x)) - c(x, y_2(x))] &\leqq e(x, y_1(x)) - e(x, y_2(x)) \\ &\leqq \lambda^2[c(x, y_1(x)) - c(x, y_2(x))]. \end{aligned} \tag{4.3}$$

By virtue of the fact that $0 < \lambda^1 < \lambda^2$, we must have $c(x, y_1(x)) \geqq c(x, y_2(x))$ for $x \in X$; otherwise, (4.3) yields a contradiction. If $c(x, y_1(x)) = c(x, y_2(x))$, then $e(x, y_2(x)) = e(x, y_1(x))$, and using the definition of $\varphi(x, \cdot)$, one may show that $y_2(x) = y_1(x)$. If $c(x, y_1(x)) > c(x, y_2(x))$, then the increasing nature of $c(x, \cdot)$ yields $y_1(x) > y_2(x)$. Thus $\varphi(x, \cdot)$ is a decreasing function for $x \in X$. Since $c(x, \cdot)$ is continuous and increasing for $x \in X$, one may show that I is decreasing. This proves (b).

To prove (c), we first show that for $x \in X$, $M(x, \cdot)$ is continuous. Choose $0 < \lambda^1 < \lambda^2 < \infty$. Observe that $\varphi(x, \lambda^1) \geqq \varphi(x, \lambda^2)$ and define

$$K(x) = \sup\{|c(x, y)| : \varphi(x, \lambda^2) \leqq y \leqq \varphi(x, \lambda^1)\} \quad \text{for } x \in X.$$

Suppose that $M(x, \lambda^1) \geqq M(x, \lambda^2)$. Since $l(x, \varphi(x, \lambda_1), \lambda_2) \leqq l(x, \varphi(x, \lambda_2), \lambda_2)$,

$$\begin{aligned}|M(x, \lambda^1) - M(x, \lambda^2)| &\leqq |l(x, \varphi(x, \lambda^1), \lambda^1) - l(x, \varphi(x, \lambda^1), \lambda^2)| \\ &\leqq \sup\{|l(x, y, \lambda^1) - l(x, y, \lambda^2)| : \varphi(x, \lambda^2) \leqq y \leqq \varphi(x, \lambda^1)\} \\ &\leqq |\lambda^1 - \lambda^2| \sup\{|c(x, y)| : \varphi(x, \lambda^2) \leqq y \leqq \varphi(x, \lambda^1)\} \\ &\leqq |\lambda^1 - \lambda^2| K(x).\end{aligned}$$

Making a similar argument when $M(x, \lambda^1) < M(x, \lambda^2)$, we show that $M(x, \cdot)$ is continuous for $x \in X$.

Fix $x \in X$ such that $M(x, \cdot)$ is continuous. Let $\lambda_i \uparrow \lambda_0$ and define $y_i = \varphi(x, \lambda_i)$ for $i = 0, 1, 2, \cdots$. Then $\{y_i\}_{i=1}^{\infty}$ is a decreasing sequence with a limit $z \in Y(x)$. Moreover, $z \geqq y_0$, and

$$\lim_{i \to \infty} l(x, y_i, \lambda_i) = \lim_{i \to \infty} M(x, \lambda_i) = M(x, \lambda_0) = l(x, y_0, \lambda_0).$$

However, by the upper semi-continuity of $e(x, \cdot)$ and continuity of $c(x, \cdot)$,

$$l(x, y_0, \lambda_0) = \lim_{i \to \infty} l(x, y_i, \lambda_i) \leqq l(x, z, \lambda_0).$$

Hence $l(x, z, \lambda_0) = l(x, y_0, \lambda_0)$, and by definition of y_0, $z = y_0$. It follows that $\varphi(x, \cdot)$ is left-continuous for $x \in X$. Since $c(x, \cdot)$ is continuous and increasing for $x \in X$, we may apply the monotone convergence theorem to complete the proof of (c).

We claim $l(x, \xi(x, \lambda), \lambda) = M(x, \lambda)$ for $x \in X$. This follows from the continuity of $M(x, \cdot)$ and $c(x, \cdot)$ along with the upper semi-continuity of $e(x, \cdot)$ for $x \in X$ as follows:

$$\begin{aligned}l(x, \xi(x, \lambda), \lambda) &\geqq \lim_{\lambda' \downarrow \lambda} l(x, \varphi(x, \lambda'), \lambda') \\ &= \lim_{\lambda' \downarrow \lambda} M(x, \lambda') = M(x, \lambda).\end{aligned}$$

Thus $l(x, \xi(x, \lambda), \lambda) = M(x, \lambda)$ for $x \in X$.

To prove (d) we use the family $\{S_\alpha : \alpha \in [0, 1]\}$ of Lemma 4.1. Fix $\lambda > 0$ and let

$$h_\alpha(x) = \begin{cases} \varphi(x, \lambda) & \text{for } x \in S_\alpha, \\ \xi(x, \lambda) & \text{for } x \in X - S_\alpha, \quad \alpha \in [0, 1]. \end{cases}$$

Note that $l(x, h_\alpha(x), \lambda) = M(x, \lambda)$ for $x \in X$. Since $\varphi(x, \lambda) \geqq \xi(x, \lambda)$ for $x \in X$, one may use (i) and (iv) of Lemma 4.1 and the monotone convergence theorem to show that $C(h_\alpha)$ is a left continuous function of α. A similar argument using (i)–(iii) of Lemma 4.1 shows that $C(h_\alpha)$ is a right-continuous (hence continuous) function of α. The existence of f in (d) now follows readily. This proves the lemma.

Theorem 4.3 below proves the existence of uniformly optimal allocation schedules. Let J be an interval of extended reals. An *allocation schedule* over J is a Borel function η defined on $X \times J$ such that for $x \in X$, $\eta(x, \cdot)$ is increasing and $\eta(x, v) \in Y(x)$ for $v \in J$. We say that an allocation schedule η is *uniformly optimal* over J if

$$E(\eta(\cdot, v)) = \max\{E(q) : C(q) \leqq v\} \quad \text{for } v \in J.$$

This definition is an extension of the definition of uniform optimality given in [1] in conjunction with search theory. The notion is a natural one for search in that one desires a search plan (allocation schedule) such that one is always achieving the maximum probability of detection from the effort expended.

In Theorem 4.3 below, one would like to prove the existence of a uniformly optimal allocation schedule over $[C(T), C(U)]$. However, this is not always possible since, for example, the range of C need not be connected.

THEOREM 4.3. *Assume Ω, e, and c are Borel. Let μ be σ-finite, nonatomic, and Borel regular. Suppose $-\infty < E(T) \leqq E(U) < \infty$, $|C(T)| < \infty$ and for $x \in X$, $Y(x)$ is compact in $\bar{\mathscr{E}}_1$, $e(x, \cdot)$ is increasing and right-continuous, and $c(x, \cdot)$ is continuous increasing, and extended real-valued. Then there exists an allocation schedule η which is uniformly optimal over $[C(T), C(V))$, where $V = \xi(\cdot, 0)$. Moreover, $E(V) = \max\{E(q) : C(q) < \infty\}$, and if $C(U) < \infty$, then*

$$E(V) = \max\{E(q) : q \in \Phi\} \tag{4.4}$$

and extending η by defining $\eta(\cdot, C(V)) = V$, one obtains an allocation schedule which is uniformly optimal over $[C(T), C(V)]$.

Proof. First we define

$$g(x) = \lim_{\lambda \uparrow \infty} \varphi(x, \lambda).$$

We claim that $c(x, g(x)) = c(x, T(x))$ for $x \in X$. To see this, we note that $\varphi(x, \lambda) \geqq g(x)$ for $0 < \lambda < \infty$, and suppose $c(x, g(x)) > c(x, T(x))$ for x in a set of positive measure. Then for such x,

$$\infty > \lambda_0 \equiv \frac{e(x, U(x)) - e(x, T(x))}{c(x, g(x)) - c(x, T(x))} \geqq \frac{e(x, \varphi(x, \lambda)) - e(x, T(x))}{c(x, \varphi(x, \lambda)) - c(x, T(x))} \quad \text{for } \lambda > 0,$$

which contradicts $l(x, \varphi(x, \lambda), \lambda) = M(x, \lambda)$ for $\lambda > \lambda_0$. Thus the claim is verified.

In addition, g is strongly optimal. To see this, we observe that $C(g) = C(T)$ and suppose that g is not strongly optimal. That is, there exists $q^* \in \Phi$ such that $E(q^*) > E(g)$ and $C(q^*) \leqq C(g)$. Since $c(x, \cdot)$ is increasing,

$$c(x, q^*(x)) = c(x, T(x)) = c(x, g(x)) \quad \text{for } x \in X. \tag{4.5}$$

Since $E(q^*) > E(g)$, there exists a set P with positive measure such that

$$e(x, q^*(x)) > e(x, g(x)) \quad \text{for } x \in P. \tag{4.6}$$

For $x \in P$, the increasing nature of $e(x, \cdot)$ yields that $q^*(x) > g(x)$. Hence we may choose λ_x such that $q^*(x) > \varphi(x, \lambda_x) > g(x)$. Then

$$e(x, \varphi(x, \lambda_x)) - \lambda_x c(x, \varphi(x, \lambda_x)) \geqq e(x, q^*(x)) - \lambda_x c(x, q^*(x)). \tag{4.7}$$

Again, the increasing nature of $c(x, \cdot)$ along with (4.5) implies that $c(x, \varphi(x, \lambda_x)) = c(x, q^*(x))$. Equation (4.7) now yields $e(x, \varphi(x, \lambda_x)) \geqq e(x, q^*(x))$. The increasing nature of $e(x, \cdot)$ implies that $e(x, \varphi(x, \lambda_x)) = e(x, q^*(x))$. By the same argument as above, we may show that $e(x, \varphi(x, \lambda)) = e(x, q^*(x))$ for $\lambda \geqq \lambda_x$. The definition of $g(x)$ and the right continuity of $e(x, \cdot)$ may now be combined to prove that $e(x, q^*(x)) = e(x, g(x))$ for $x \in P$, which contradicts (4.6). Thus $E(q^*) \leqq E(g)$ and

g is strongly optimal. Define

$$\eta(x, C(T)) = g(x) \quad \text{for } x \in X.$$

Now I is monotone so it has only a countable number of discontinuities. Let N be a countable index set such that $\{\lambda_n : n \in N\}$ is the set of discontinuity points of I. Let $J_n = [C(\xi(\cdot, \lambda_n)), I(\lambda_n)]$ for $n \in N$. Then the intervals J_n are disjoint and are the jump intervals at the discontinuity points of I. Note that $\lim_{\lambda \downarrow 0} I(\lambda) = C(V)$. For $v \in (C(T), C(V)) - \bigcup_{n \in N} J_n$, let

$$\lambda^*(v) = \sup \{\lambda : I(\lambda) = v\}.$$

By the left continuity of I, $I(\lambda^*(v)) = v$. For $v \in J_n$, let $\lambda^*(v) = \lambda_n$. By (d) of Lemma 4.2, we may find a function f_n defined on $X \times J_n$ such that $f_n(x, \cdot)$ is increasing for $x \in X$ and for $v \in J_n$, $f_n(\cdot, v) \in \Phi$, $C(f_n(\cdot, v)) = v$, and

$$l(x, f_n(x, v), \lambda_n) = M(x, \lambda_n) \quad \text{for } x \in X.$$

We now define for $x \in X$,

$$\eta(x, v) = \begin{cases} \varphi(x, \lambda^*(v)) & \text{if } v \in (C(T), C(V)) - \bigcup_{n \in N} J_n, \\ f_n(x, v) & \text{if } v \in J_n \quad \text{for some } n \in N. \end{cases}$$

Then for $v \in (C(T), C(V))$, $C(\eta(\cdot, v)) = v$ and $\eta(\cdot, v)$ satisfies the pointwise multiplier rule with $\lambda = \lambda^*(v) \geqq 0$. Thus $\eta(\cdot, v)$ is strongly optimal by Theorem 2.1 of [8].

To prove that $\eta(x, \cdot)$ is increasing for $x \in X$, we let v and s be such that $C(T) < v < s < C(V)$. Then there exists a set P such that $\mu(P) > 0$ and

$$c(x, \eta(x, s)) > c(x, \eta(x, v)) \quad \text{for } x \in P.$$

Since (3.7) holds with λ^1, λ^2, q_1, and q_2 replaced by $\lambda(v)$, $\lambda(s)$, $\eta(\cdot, v)$ and $\eta(\cdot, s)$ respectively, $\lambda^*(s) \leqq \lambda^*(v)$. Suppose $\lambda^*(s) < \lambda^*(v)$. Then $\eta(x, s) \geqq \xi(x, \lambda^*(s)) \geqq \varphi(x, \lambda^*(v)) \geqq \eta(x, v)$ for $x \in X$. If $\lambda^*(s) = \lambda^*(v)$, then v and s are both in the same J_n for some $n \in N$ and $\eta(x, s) \geqq \eta(x, v)$ for $x \in X$ by construction. Since $\eta(x, C(T)) = g(x) \leqq \varphi(x, \lambda)$ for $x \in X$ and $\lambda > 0$, we have $\eta(x, \cdot)$ is increasing for $x \in X$. Thus η is uniformly optimal over $[C(T), C(V))$.

Suppose $q \in \Phi$ such that $|C(q)| < \infty$. Then $|c(x, q(x))| < \infty$ for $x \in X$, and we have

$$\begin{aligned} e(x, V(x)) &\geqq \lim_{\lambda \downarrow 0} e(x, \varphi(x, \lambda)) - \lambda c(x, \varphi(x, \lambda)) \\ &\geqq \lim_{\lambda \downarrow 0} e(x, q(x)) - \lambda c(x, q(x)) = e(x, q(x)) \quad \text{for } x \in X. \end{aligned}$$

Thus $E(V) = \max \{E(q) : |C(q)| < \infty\}$. If $C(U) < \infty$, then (4.4) follows, and by setting $\eta(x, C(V)) = V(x)$ for $x \in X$, the theorem follows.

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