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NECESSARY AND SUFFICIENT CONDITIONS FOR OPTIMAL CONTROL OF SEMI-MARKOV JUMP PROCESSES*

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Abstract. A class of controlled semi-Markov jump processes is defined in this paper. Conditions are found which guarantee that satisfaction of the dynamic programming equations for stochastic control is necessary and sufficient for the minimization of the expected discounted cost of a controlled semi-Markov process over a random time. Terminal costs are included, and the controls are allowed to depend on both the present state of the process and the length of time the process has been in that state.

1. Introduction. The problem under consideration in this paper is that of finding necessary and sufficient conditions for a control to minimize the expected discounted cost of a controlled semi-Markov process over a random time. The setting of our results is more general than is usually considered for control problems involving semi-Markov processes in that we allow the control to depend on the present state of the process and the length of time the process has been in that state. Moreover, we allow terminal costs and do not restrict ourselves to finite state spaces.

Our approach is to consider a semi-Markov process as a two-dimensional Markov process and to find an equation (i.e., (4.9)) involving the infinitesimal generator of this two-dimensional process. Under specified conditions, we show that satisfaction of this equation is a necessary and sufficient condition for a control to be optimal (i.e., to minimize cost). Equation (4.9) is a form of the dynamic programming conditions for stochastic control problems. In order to obtain our result, we define the notion of a controlled semi-Markov process in § 2 and compute its infinitesimal generator in § 3. Section 4 contains the results on necessity and sufficiency. These results are also specialized to the case of undiscounted cost and to Markov jump process.

The principle of dynamic programming has been applied to the optimal control of discrete time Markov processes by many authors, e.g., Howard [9], Derman [4] and [5], Blackwell [2], Astrom [1] and Ross [15]. In the case of continuous time Markov processes, most of the work on optimal stochastic control has dealt with diffusion processes. A review of the literature on optimal control of diffusion processes is given in [7] where there is a discussion of the dynamic programming conditions for optimal control of diffusion processes. These conditions involve the infinitesimal generator of the diffusion and are the analogue, for stochastic control, of Bellman's principle of optimality. Although there are results showing that these equations are necessary or sufficient for the optimal control of certain classes of diffusion processes, very few results of this type exist for continuous time Markov processes which are not diffusions or for non-Markovian processes. Some exceptions to this are to be found in [14] and in a remark on page 527 of [12]. Theorem 4.5 of this paper extends the results on the necessity and sufficiency of the dynamic programming conditions for optimal control to semi-Markov jump processes.

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Optimal control of semi-Markov processes has been studied in [3], [10], [11], [13] and [16]. However, these references deal with a more restricted class of processes and a more restricted class of controls than considered here. In particular, all of these references except [16] restrict their attention to processes which have only a finite number of states, and all except [3] restrict the class of controls to those which depend only on the state of the process and not on the length of time the process has remained in that state. In addition the control problem which we consider involves terminal conditions while the references mentioned in this paragraph deal with stationary problems, for the most part.

2. Controlled semi-Markov jump processes. Let $\{X_t : 0 \leqq t < \infty\}$ be a semi-Markov jump process (see [17] for definition) defined on the probability space $(\Omega, \mathscr{A}, P)$. In this paper we make the slight generalization that we allow the state space Σ of $\{X_t\}$ to be an arbitrary set with a topology $\mathscr{T}_1$. Let $\mathscr{S}_1$ be the Borel σ-field formed from $\mathscr{T}_1$. We assume that $\mathscr{S}_1$ contains all singleton subsets of Σ. (The need for $\mathscr{S}_1$ to contain all singletons will be made clear later.) Let R^+ be the nonnegative real numbers and $\mathscr{T}_2$ the usual topology on R^+. Let $\mathscr{S}_2$ be the set of Borel sets of R^+. Let Y_0 be a nonnegative random variable defined on $(\Omega, \mathscr{A}, P)$ and for $t \geqq 0$, define

$$\nu_t = \inf\{s > t : X_{t+s} \neq X_t\},$$

$$Y_t = \begin{cases} Y_0 + t & \text{for } t < \nu_0, \\ t - \sup[s : 0 \leqq s \leqq t \text{ and } X_s \neq X_t] & \text{for } t \geqq \nu_0. \end{cases}$$

For convenience of notation, we denote ν_0 by ν and let $Z_t = (X_t, Y_t)$ for $t \geqq 0$. In [17], ν is required to be finite with probability one. In this paper we do not make that requirement.

We use $P_{x,s}$ and $E_{x,s}$ to denote, respectively, probability and expectation conditioned on $(X_0, Y_0) = (x, s)$. Define

$$\hat{a}(x, s) = P_{x,0}[\nu \leqq s] \qquad \text{for } x \in \Sigma \text{ and } s \in R^+,$$

$$\hat{k}(x, s, \Gamma) = P_{x,0}[X_\nu \in \Gamma | \nu = s] \quad \text{for } (x, s) \in \Sigma \times R^+ \text{ and } \Gamma \in \mathscr{S}_1.$$

Then $\hat{a}$ and $\hat{k}$ give the jump time and after jump distributions of the semi-Markov process. Moreover,

$$P_{x,0}[X_\nu \in \Gamma] = \int_0^\infty \hat{k}(x, s, \Gamma)\hat{a}(x, ds).$$

One may think of the functions $\hat{a}$ and $\hat{k}$ as specifying the semi-Markov process. This is the point of view we adopt for control problems.

We suppose that for each $x \in \Sigma$ there is a family of measures on $\mathscr{S}_1$,

$$\{k(x, b, \cdot) : b \in B\},$$

where B is some index set. We assume that B is endowed with a topology and that $\mathscr{S}_3$ is the corresponding Borel σ-field. A control u is an ordered pair of functions (γ, β) with

$$\gamma : \Sigma \times R^+ \to R^+ \quad \text{and} \quad \beta : \Sigma \times R^+ \to B$$

such that:

(i) γ and β are Borel measurable (i.e., $\mathcal{S}_1 \times \mathcal{S}_2$ measurable where $\mathcal{S}_1 \times \mathcal{S}_2$ denotes the product σ-field generated by $\mathcal{S}_1$ and $\mathcal{S}_2$),

(ii) for $x \in \Sigma$, there exists $t(x) > 0$ such that $\int_0^{t(x)} \gamma(x, s)\, ds < \infty$,

(iii) $k(\cdot, \beta, \Gamma)$ is Borel measurable for each $\Gamma \in \mathcal{S}_1$.

Let U be the class of controls which satisfy (i)–(iii). A control $u = (\gamma, \beta) \in U$ determines a semi-Markov process $\{X_t^u\}$ in the following manner. Let

$$a_\gamma(x, t) = 1 - \exp\left(-\int_0^t \gamma(x, s)\, ds\right) \quad \text{for } (x, t) \in \Sigma \times R^+.$$

Suppose that $(X_0^u, Y_0^u) = (x, s)$. Then at time 0, the process starts in state x and remains there a random time ξ_1, such that

$$P_{x,0}\{\xi_1 \leqq t\} = \frac{a_\gamma(x, s + t) - a_\gamma(x, s)}{(1 - a_\gamma(x, s))}.$$

By (ii) we may define the sample paths of $\{X_t^u\}$ to be right continuous. That is at time $\xi_1 > 0$, the process transitions to the state $X_{\xi_1}^u$, where

$$P_{x,0}\{X_{\xi_1}^u \in \Gamma\} = k(x, \beta(x, \xi_1), \Gamma).$$

The process stays in state $X_{\xi_1}^u$ for a random time $\xi_2 > 0$ such that $P_{x,0}\{\xi_2 \leqq t\} = a_\gamma(X_{\xi_1}^u, t)$ and at time $\xi_1 + \xi_2$ transitions to $X_{\xi_1+\xi_2}^u$, where

$$P_{x,0}\{X_{\xi_1+\xi_2}^u \in \Gamma\} = k(X_{\xi_1}^u, \beta(X_{\xi_1}^u, \xi_2), \Gamma).$$

The process $\{X_t^u\}$ continues in this way transitioning to a new state after remaining a random time in the present state.

For $\omega \in \Omega$ and $u \in U$, let $\zeta(\omega)$ be the first cluster point of the jump times of the sample path $\{X_t^u(\omega): t \geqq 0\}$. If there is no cluster point, we let $\zeta(\omega) = \infty$. We call ζ the terminal time of the process $\{X_t^u\}$. The process $\{Z_t^u\} = \{X_t^u, Y_t^u\}$ with the terminal time ζ is a two-dimensional Markov process with stationary Borel measurable transition probabilities.

THEOREM 2.1. *If $u \in U$, then $\{X_t^u\}$ is a semi-Markov jump process, and $\{Z_t^u\}$ is a strongly measurable strong Markov process.*

Proof. In order to satisfy the definition of a semi-Markov process, we must show that $\{Z_t^u\}$ is a strong Markov process with right-continuous sample paths. Having shown this, it follows automatically that $\{Z_t^u\}$ is strongly measurable (see [6, vol. I, p. 98]). By our definition of $\{X_t^u\}$, the sample paths of $\{Z_t^u\}$ are right continuous in the $\mathcal{T}_1 \times \mathcal{T}_2$ topology regardless of the topology $\mathcal{T}_1$ on Σ. In particular, one may choose $\mathcal{T}_1 = \mathcal{D}$, the discrete topology on Σ.

Since $\mathcal{S}_1$ contains all the singleton subsets of Σ, it follows that $\{Z_t^u\}$ is a right-continuous Markov process on the topological measurable space $(\Sigma \times R^+, \mathcal{D} \times \mathcal{T}_2, \mathcal{S}_1 \times \mathcal{S}_2)$ (see [6, vol. II, p. 222] for the definition of the topological measurable space).

For $t \geqq 0$, let

$$\rho_t^u(x, s, \Gamma, r) = P_{x,s}[X_t^u \in \Gamma; Y_t^u \leqq r] \quad \text{for } \Gamma \in \mathcal{S}_1 \quad \text{and} \quad r \geqq 0.$$

Let F be the set of bounded measurable (with respect to $\mathscr{S}_1 \times \mathscr{S}_2$) functions $f:\Sigma \times R^+ \to R$, where R is the real numbers. For $f \in F$, define

$$\|f\| = \sup_{(x,s)\in\Sigma\times R^+} |f(x,s)|,$$

$$\mathbf{T}_t^u f(x,s) = \int_{\Sigma\times R^+} f(y,r)\rho_t^u(x,s,dy,dr).$$

In order to use Theorem 3.10 of [6] to prove our theorem, we show that for each $t \in R^+$, T_t^u maps functions $f \in F$ which are continuous in the $\mathscr{D} \times \mathscr{T}_2$ topology into continuous functions. To prove this, it is sufficient to show that for each $(x,s) \in \Sigma \times R^+$ and $t > 0$,

$$\lim_{h\to 0} \mathbf{T}_t^u f(x,s) - \mathbf{T}_t^u f(x,s+h) = 0$$

whenever $f(x,\cdot)$ is continuous in the $\mathscr{T}_2$ topology. Let $t > 0$ and fix (x,s). Consider the case in which $(X_0, Y_0) = (x,s)$. Then with probability

$$[1 - a_y(x,s+h)]/[1 - a_y(x,s)],$$

$(X_h^u, Y_h^u) = (x, s+h)$, where $h > 0$. Thus for $h > 0$,

$$\rho_{t+h}^u(x,s,\Gamma,r) = \frac{1 - a_y(x,s+h)}{1 - a_y(x,s)}\rho_t^u(x,s+h,\Gamma,r) + \frac{a_y(x,s+h) - a_y(x,s)}{1 - a_y(x,s)}G_h(\Gamma,r),$$

where $G_h(\Gamma,r)$ gives the probability that $X_t \in \Gamma$ and $Y_t \leqq r$ given that a jump occurred in the interval $[0,h]$. Solving the above equation for $\rho_t^u(x,s+h,\Gamma,r)$, it follows that

$$\begin{aligned}\mathbf{T}_t^u f(x,s+h) &= \frac{1 - a_y(x,s)}{1 - a_y(x,s+h)}\mathbf{T}_{t+h}^u f(x,s) \\ &\quad - \frac{a_y(x,s+h) - a_y(x,s)}{1 - a_y(x,s+h)}\int_{\Sigma\times R^+} f(y,r)G_h(dy,dr) \\ &= \mathbf{T}_{t+h}^u f(x,s) + w(h),\end{aligned}$$

where $w(h) \to 0$ as $h \to 0+$. Hence, for $h > 0$,

$$\begin{aligned}|\mathbf{T}_t^u f(x,s) - \mathbf{T}_t^u f(x,s+h)| &\leqq |\mathbf{T}_t^u f(x,s) - \mathbf{T}_{t+h}^u f(x,s)| + o(1) \\ &\leqq |E_{x,s}[f(X_t^u, Y_t^u) - f(X_{t+h}^u, Y_{t+h}^u)]| + o(1).\end{aligned} \tag{2.1}$$

By the right continuity of $\{(X_t^u, Y_t^u)\}$ and the continuity of $f(x,\cdot)$ for $x \in \Sigma$, it follows that $\lim_{h\to 0+} f(X_{t+h}^u, Y_{t+h}^u) = f(X_t^u, Y_t^u)$ a.s. Thus, the dominated convergence theorem and (2.1) yield that $\lim_{h\to 0+} |\mathbf{T}_t^u f(x,s) - \mathbf{T}_t^u f(x,s+h)| = 0$.

To show that $\lim_{h\to 0+} |\mathbf{T}_t^u f(x,s) - \mathbf{T}_t^u f(x,s-h)| = 0$ for $s > 0$, one observes that

$$\rho_t^u(x,s-h,\Gamma,r) = \frac{1 - a_y(x,s)}{1 - a_y(x,s-h)}\rho_{t-h}^u(x,s,\Gamma,r) + \frac{a_y(x,s) - a_y(x,s-h)}{1 - a_y(x,s-h)}G_h(\Gamma,r),$$

and that

$$|\mathbf{T}_t^u f(x,s) - \mathbf{T}_t^u f(x,s-h)| \leqq |E_{x,s}[f(X_t^u, Y_t^u) - f(X_{t-h}^u, Y_{t-h}^u)]| + o(1).$$

Note that

$$\lim_{h\to 0+} (X_{t-h}(\omega), Y_{t-h}(\omega)) \to (X_t(\omega), Y_t(\omega)) \tag{2.2}$$

unless there is a jump at time t. Since the probability of having a jump at time t is 0, (2.2) holds for a.e. $\omega \in \Omega$. For each ω for which (2.2) holds, we have $X_{t-s}^u(\omega) = X_t^u(\omega)$ for $t - Y_t^u(\omega) \leqq s \leqq t$. Since $f(x,\cdot)$ is continuous for $x \in \Sigma$, it follows that $f(X_{t-h}^u, Y_{t-h}^u) \to f(X_t^u, Y_t^u)$ a.s. As before, we use the dominated convergence theorem to conclude that $\lim_{h\to 0+} |\mathbf{T}_t^u f(x,s) - \mathbf{T}_t^u f(x,s-h)| = 0$. It follows that $T_t^u f$ is continuous in the $\mathscr{D} \times \mathscr{T}_2$ topology.

By what we have just shown, $\{Z_t^u\}$ is a Feller process on the topological measure space $(\Sigma \times R^+, \mathscr{D} \times \mathscr{T}_2, \mathscr{S}_1 \times \mathscr{S}_2)$, and by Theorem 3.10 of [6], it is strong Markov. This proves the theorem.

We call $\{X_t^u\}$ a *controlled semi-Markov jump process.*

We suppose there is a cost rate $c(x, s, u(x, s))$ associated with being in state x for time s when using control $u(x, s)$. When convenient, we shall use the variable z to represent a point $(x, s) \in \Sigma \times R^+$. Thus, $c(z, u(z))$ will often be used in place of $c(x, s, u(x, s))$. In addition we suppose there is a function $C: \Sigma \times R^+ \to R$ such that $C(z)$ gives the cost of stopping the process in state z. Let τ be a stopping time of $\{X_t^u\}$. Fix $\lambda > 0$ and define for $u \in U$,

$$\varphi_u(z) = E_z\left[\int_0^\tau e^{-\lambda t} c(Z_t^u, u(Z_t^u))\,dt + e^{-\lambda\tau} C(Z_\tau^u)\right], \qquad z \in \Sigma \times R^+. \tag{2.3}$$

For the case of undiscounted cost, we define for $u \in U$,

$$\hat{\varphi}_u(z) = E_z\left[\int_0^\tau c(Z_t^u, u(Z_t^u))\,dt + C(Z_\tau^u)\right], \qquad z \in \Sigma \times R^+. \tag{2.4}$$

In order that the above integrals be well-defined, we assume that c and C are Borel measurable. To allow for the possibility that $\tau(\omega) = \zeta(\omega)$ for some ω we adjoin a state z_0 to $\Sigma \times R^+$ and define $Z_\zeta = z_0$. The function C is assumed to be defined on this extended state space.

The dependence of τ on u will be suppressed. Let U' be a specified set of controls. We are interested in controls $u^* \in U'$ such that

$$\varphi_{u^*}(x, s) = \min_{u\in U'} \varphi_u(x, s), \qquad (x, s) \in \Sigma \times R^+. \tag{2.5}$$

Such a control is called *optimal in* U'. In order to find necessary and sufficient conditions for a function φ_{u^*} to satisfy (2.5), we discuss the infinitesimal operator of a semi-Markov jump process.

3. Infinitesimal operator of a semi-Markov jump process. If $\{X_t^u\}$ is a controlled semi-Markov jump process, then the process $\{(X_t^u, Y_t^u)\}$ is a two-dimensional, right-continuous Markov process with stationary transition probabilities. Thus, we may define $\mathbf{A}^u$, the weak infinitesimal operator for this process, in the manner given in [6]. Let $\tilde{F}$ be the set of all $f \in F$ such that

$$\lim_{t \to 0+} \mathbf{T}_t^u f(x, s) = f(x, s) \quad \text{for } (x, s) \in \Sigma \times R^+.$$

The weak infinitesimal operator is defined by

$$\mathbf{A}^u f = \lim_{t \to 0+} (\mathbf{T}_t^u f - f)/t,$$

whenever the limit of the right side exists in the weak sense and is a member of $\tilde{F}$. We say that $\lim_{n \to \infty} f_n = f$ in the weak sense if:

(a) $\lim_{n \to \infty} f_n(x, s) = f(x, s)$ for each $(x, s) \in \Sigma \times R^+$, and

(b) $\|f_n\|$, $n = 1, 2, \cdots$, are bounded.

Let $f^+(x, \cdot)$ denote the right-hand derivative of $f(x, \cdot)$. If $\gamma(x, \cdot)$ is right continuous at s, then

$$\gamma(x, s) = \frac{a_\gamma^+(x, s)}{1 - a_\gamma(x, s)}.$$

In this case $\gamma(x, s)$ is the jump rate of the process $\{X_t^u\}$ given that it has been in state x for a time s. Recall that $k(x, \beta(x, s), \Gamma)$ gives the probability, when using control $\beta(x, s)$, that the transition from x will be into the set Γ given that the transition takes place after being in x for a time s. Thus if $\gamma(x, \cdot)$ is right continuous, it follows from [8, § 411] that

$$a_\gamma(x, t) = \int_0^t a_\gamma^+(x, s)\, ds$$

and

$$P_{x,0}\{X_v^u \in \Gamma \text{ and } v \leq t\} = \int_0^t k(x, \beta(x, s), \Gamma)\gamma(x, s) \exp\left(-\int_0^s \gamma(x, r)\, dr\right) ds.$$

We now compute $\mathbf{A}^u f$ for a particular class of functions f and controlled semi-Markov jump processes.

LEMMA 3.1. *Let $u = (\gamma, \beta) \in U$ be such that for $x \in \Sigma$:*

(i) *$\gamma(x, \cdot)$ is right continuous and $\gamma \leq M \in R^+$,*

(ii) *$\beta(x, \cdot)$ is piecewise constant, right continuous, and has only a finite number of discontinuities.*

Suppose $f \in F$ is such that f^+ exists and is bounded, and $f^+(x, \cdot)$ is right continuous for all $x \in \Sigma$. Then

$$\mathbf{A}^u f(x, s) = f^+(x, s) + \gamma(x, s)\left[\int_\Sigma f(y, 0) k(x, \beta(x, s), dy) - f(x, s)\right] \tag{3.1}$$

for $(x, s) \in \Sigma \times R^+$.

Proof. Since $\gamma(x, s) = a_\gamma^+(x, s)/[1 - a(x, s)]$, it follows that $\gamma \leq M$ implies $a_\gamma(x, h) \leq Mh$ and

$$\frac{a_\gamma^+(x, s + r)}{1 - a_\gamma(x, s)} \leq \frac{a_\gamma^+(x, s + r)}{1 - a_\gamma(x, s + r)} \leq M. \tag{3.2}$$

Thus,

$P_{x,s}$ [more than one jump in $[0,h]$]

$$= \int_0^h \int_\Sigma \frac{a_y^+(x, s + r)}{1 - a_y(x, s)} a_y(y, h - r) k(x, \beta(x, s + r), dy)\, dr$$

$$\leqq \int_0^h M^2(h - r)\, dr \leqq M^2 h^2,$$

and the probability of having more than one jump in $[0, h]$ is $o(h)$ as $h \to 0$. For r small enough, $\beta(x, s + r) = \beta(x, s)$. Thus for small $h > 0$,

$$\mathbf{T}_h^u f(x, s) - f(x, s)$$

$$= \frac{1 - a_y(x, s + h)}{1 - a_y(x, s)} f(x, s + h) + \int_0^h \int_\Sigma f(y, h - r)(1 - a_y(y, h - r))$$

$$\cdot \frac{a_y^+(x, s + r)}{1 - a_y(x, s)} k(x, \beta(x, s), dy)\, dr - f(x, s) + o(h)$$

$$= \frac{(1 - a_y(x, s + h)) f(x, s + h) - (1 - a_y(x, s)) f(x, s)}{1 - a_y(x, s)}$$

$$+ \int_\Sigma \int_0^h f(y, h - r)(1 - a_y(y, h - r)) \frac{a_y^+(x, s + r)}{1 - a_y(x, s)}\, dr\, k(x, \beta(x, s), dy) + o(h).$$

By the dominated convergence theorem,

$$\lim_{h \to 0+} \frac{1}{h}(\mathbf{T}_h^u f(x, s) - f(x, s))$$

$$= f^+(x, s) - \frac{a_y^+(x, s)}{1 - a_y(x, s)} f(x, s) + \int_\Sigma \frac{a_y^+(x, s)}{1 - a_y(x, s)} f(y, 0) k(x, \beta(x, s), dy)$$

$$= f^+(x, s) + \gamma(x, s)\left[\int_\Sigma f(y, 0) k(x, \beta(x, s), dy) - f(x, s)\right].$$

The boundedness of f and γ guarantee that $\|\mathbf{T}_h^u f - f\|/h$ is uniformly bounded as $h \to 0$. Thus the above limit holds in the weak sense. Observe that any function $g \in F$ such that $g(x, \cdot)$ is right continuous for all $x \in \Sigma$ is contained in $\tilde{F}$. Thus the right-hand side of (3.1) is in $\tilde{F}$, and the lemma is proved.

4. Necessary and sufficient conditions for optimal control of semi-Markov jump processes. In this section we prove the main result of the paper, Theorem 4.5, which gives conditions under which (4.9) is necessary and sufficient for optimality of a control u^* within a specified class of controls. Throughout this section, we let Δ be a subset of $\Sigma \times R^+$ such that $S(x) = \{s : (x, s) \in \Delta\}$ is a closed right half-line for $x \in \Sigma$. Moreover, we define

$$\pi(\Delta) = \{x : (x, s) \in \Delta \text{ for some } s \in R^+\},$$

$$\tau = \min\{\zeta, \inf[t : t \geqq 0 \text{ and } Z_t^u \in \Delta]\}.$$

We suppress the dependence of τ on u. Note that τ is a Markov or stopping time of $\{Z_t^u\}$.

THEOREM 4.1. *Let $U' \subset U$. Suppose that there exists $f \in F$ such that f is in the domain of $\mathbf{A}^u$ for $u \in U'$ and*

$$\lambda f(z) = \inf_{u \in U'} \{\mathbf{A}^u f(z) + c(z, u(z))\} \quad \text{for } z \in \Sigma \times R^+ - \Delta, \tag{4.1}$$

$$f(z) = C(z) \quad \text{for } z \in \Delta. \tag{4.2}$$

Then

$$f(z) \leqq \inf_{u \in U'} \varphi_u(z) \quad \text{for } z \in \Sigma \times R^+. \tag{4.3}$$

Proof. For f in the domain of $\mathbf{A}^u$, Theorem 1.7 of [6] yields that

$$f = \mathbf{R}_\lambda^u(\lambda f - \mathbf{A}^u f),$$

where

$$\mathbf{R}_\lambda^u f(z) = \int_0^\infty e^{-\lambda t} \mathbf{T}_t f(z)\, dt \quad \text{for } z \in \Sigma \times R^+.$$

Thus, by Theorem 5.1 of [6],

$$E_z[e^{-\lambda\tau} f(Z_\tau^u)] - f(z) = E_z\left[\int_0^\tau e^{-\lambda t}[\mathbf{A}^u f(Z_t^u) - \lambda f(Z_t^u)]\, dt\right]. \tag{4.4}$$

For $z \in \Delta$, (4.3) follows trivially from the fact that $\varphi^u(z) = C(z)$ for $z \in \Delta$ and $u \in U$. Let $u \in U'$. By (4.1),

$$c(z, u(z)) + A^u f(z) - \lambda f(z) \geqq 0 \quad \text{for } z \in \Sigma \times R^+ - \Delta. \tag{4.5}$$

Integrating the left-hand side of (4.5), we obtain

$$E_z\left[\int_0^\tau e^{-\lambda r} c(Z_r^u, u(Z_r^u))\, dr + \int_0^\tau e^{-\lambda r}[\mathbf{A}^u f(Z_r^u) - \lambda f(Z_r^u)]\, dr\right] \geqq 0$$
$$\text{for } z \in \Sigma \times R^+ - \Delta.$$

By (4.4) and (4.2), we obtain

$$E_z\left[\int_0^\tau e^{-\lambda r} c(Z_r^u, u(Z_r^u))\, dr + e^{-\lambda\tau} C(Z_\tau^u)\right] \geqq f(z) \quad \text{for } z \in \Sigma \times R^+ - \Delta.$$

Since u is an arbitrary member of U', we have $\varphi_u(z) \geqq f(z)$ for $u \in U'$ and $z \in \Sigma \times R^+ - \Delta$. The theorem follows.

For the undiscounted case we prove the following corollary.

COROLLARY 4.2. *Let $U' \subset U$. Suppose there exists $f \in F$ such that f is in the domain of $\mathbf{A}^u$ for $u \in U'$ and*

$$0 = \inf_{u \in U'} \{\mathbf{A}^u f(z) + c(z, u(z))\} \quad \text{for } z \in \Sigma \times R^+ - \Delta,$$

$$f(z) = C(z) \quad \text{for } z \in \Delta.$$

If $E_z[\tau] < \infty$ *for* $z \in \Sigma \times R^+$, *then*

$$f(z) \leqq \inf_{u \in U'} \hat{\varphi}_u(z) \quad \text{for } z \in \Sigma \times R^+.$$

Proof. The corollary follows in the same manner as Theorem 4.1 by the use of the corollary to Theorem 5.1 of [6].

In Theorem 4.3 below, we find a necessary condition for optimality which we shall use in the proof of our main theorem. The operator $\mathbf{Q}$ used in the proof of Theorem 4.3 is a generalization to continuous time processes of the operator T used by Blackwell in [2] for discrete time processes.

Fix h such that $0 < h \leqq \infty$ and define the following stopping times:

$$\sigma_1(x, s) = \min\{h, \delta_1, \tau\},$$

where δ_1 gives the waiting time for the first jump of the process $\{X_t\}$ given $(X_0, Y_0) = (x, s)$. Although σ_1 and δ_1 depend on the control u, we shall not indicate this dependence. In a similar fashion, we define

$$\sigma_n(Z_{\sigma_{n-1}}) = \min\{h, \delta_n, \tau\} \quad \text{for } n \geqq 2,$$

where δ_n is the waiting time for the first jump of the process after σ_{n-1}. For convenience of notation, we shall write σ_n for $\sigma_n(Z_{\sigma_{n-1}})$ and let

$$\xi_0 = 0, \qquad \xi_n = \sigma_1 + \cdots + \sigma_n \qquad \text{for } n \geqq 1.$$

Define

$$W(x, s, u) = E_{x,s} \int_0^{\sigma_1} e^{-\lambda t} c(x, s + t, u(x, s + t))\, dt \quad \text{for } u \in U, (x, s) \in \Sigma \times R^+.$$

In addition, we let

$$H_u(x, s, \Lambda, r) = P_{x,s}[Z^u_{\sigma_1} \in \Lambda \text{ and } \sigma_1 \leqq r],$$

where $(x, s) \in \Sigma \times R^+$, Λ is a measurable subset of $\Sigma \times R^+$, $r \geqq 0$, and $u \in U$. Let $U' \subset U$ and let u^* be optimal in U'. Define

$$\psi(x, s, U') = \inf_{u \in U'} \left\{ W(x, s, u) + \int e^{-\lambda r} \varphi_{u^*}(z) H_u(x, s, dz, dr) \right\}$$
$$\text{for } (x, s) \in \Sigma \times R^+,$$

where an integral without an indicated range is understood to run over $\Sigma \times R^+ \times R^+$.

We say that $U' \subset U$ is closed under one point exchanges if $u_1, u_2 \in U'$ imply

$$u(y, r) \equiv \begin{cases} u_1(y, r) & \text{for } y = x \text{ and } r \geqq s, \\ u_2(y, r), & \text{otherwise}, \end{cases}$$

is a member of U', where $(x, s) \in \Sigma \times R^+$ is arbitrary. Since singleton subsets of Σ are measurable, U is closed under one point exchanges. Similarly, the following sets are closed under one point exchanges:

$$U_1 = \{u : u = (\gamma, \beta) \in U, \gamma(x, \cdot) \text{ is right continuous for } x \in \Sigma \text{ and } \gamma \leqq M\},$$

$$U_2 = \{u : u \in U \text{ and } c(x, \cdot, u(x, \cdot)) \text{ is right continuous for } x \in \Sigma\},$$

$$U_1 \cap U_2,$$

where M is a real number and $c(x, \cdot, u(x, \cdot))$ is understood to be a function defined on R^+.

THEOREM 4.3. *Suppose $U' \subset U$ is closed under one point exchanges. Let C and c be bounded, and let $\tau(\omega) < \zeta(\omega)$ for a.e. $\omega \in \Omega$ such that $\zeta(\omega) < \infty$. If u^* is optimal in U', then*

$$\varphi_{u^*}(z) = \psi(z, U') \quad \text{for } z \in \Sigma \times R^+. \tag{4.6}$$

Proof. We observe that for $(x, s) \in \Sigma \times R^+$,

$$\varphi_{u^*}(x, s) = W(x, s, u^*) + \int e^{-\lambda r} \varphi_{u^*}(z) H_{u^*}(x, s, dz, dr),$$

and thus $\varphi_{u^*}(x, s) \geqq \psi(x, s, U')$.

Suppose that for some $(x', s') \in \Sigma \times R^+$, $\varphi_{u^*}(x', s') > \psi(x', s', U')$. Then let $v \in U'$ be such that

$$\varphi_{u^*}(x', s') > W(x', s', v) + \int e^{-\lambda r} \varphi_{u^*}(z) H_v(x', s', dz, dr). \tag{4.7}$$

Define

$$\hat{u}(y, r) = \begin{cases} v(y, r) & \text{for } y = x' \text{ and } r \geqq s', \\ u^*(y, r), & \text{otherwise.} \end{cases}$$

Since U' is closed under one point exchanges, $\hat{u} \in U'$. For $f \in F$, define the operator $\mathbf{Q} : F \to F$ as follows:

$$\mathbf{Q}f(z) = W(z, \hat{u}) + \int e^{-\lambda r} f(z') H_{\hat{u}}(z, dz', dr) \quad \text{for } z \in \Sigma \times R^+.$$

Let $\mathbf{Q}^1 = \mathbf{Q}$, and define $\mathbf{Q}^{n+1} f = \mathbf{Q}(\mathbf{Q}^n f)$ for $n \geqq 1$. Then we have

$$\begin{aligned} \mathbf{Q}^2 \varphi_{u^*}(x, s) &= W(x, s, \hat{u}) + \int e^{-\lambda r} \mathbf{Q} \varphi_{u^*}(z) H_{\hat{u}}(x, s, dz, dr) \\ &= W(x, s, \hat{u}) + E_{x,s}[e^{-\lambda \xi_1} W(Z^{u}_{\xi_1}, \hat{u})] + E_{x,s}[e^{-\lambda \xi_2} \varphi_{u^*}(Z^{\hat{u}}_{\xi_2})] \end{aligned}$$

and

$$\mathbf{Q}^n \varphi_{u^*}(x, s) = E_{x,s}\left[\sum_{j=1}^{n} \exp(-\lambda \xi_{j-1}) W(Z^{\hat{u}}_{\xi_{j-1}}, \hat{u})\right] + E_{x,s}[\exp(-\lambda \xi_n) \varphi_{u^*}(Z^{\hat{u}}_{\xi_n})]. \tag{4.8}$$

Note that $\mathbf{Q}^n \varphi_{u^*}$ gives the cost of using $\hat{u}$ up to time ξ_n plus the terminal cost given by the second term on the right of (4.8). Observe that as $n \to \infty$, the first term on the right-hand side of (4.8) approaches

$$E_{x,s} \int_0^{\tau} e^{-\lambda t} c(Z^{\hat{u}}_t, \hat{u}(Z^{\hat{u}}_t))\, dt.$$

If $\tau(\omega) < \infty$, then there exists $N(\omega)$ such that for $n \geqq N(\omega)$, $\xi_n(\omega) = \tau(\omega)$, by virtue of the assumption that $\tau(\omega) < \zeta(\omega)$ whenever $\zeta(\omega) < \infty$. This combined with the boundedness of C and c gives

$$\lim_{n\to\infty} e^{-\lambda\xi_n(\omega)}\varphi_{u^*}(Z^{\hat{u}}_{\xi_n}(\omega)) = \begin{cases} e^{-\lambda\tau(\omega)}C(Z^{\hat{u}}_{\tau}(\omega)) & \text{for a.e. } \omega \in \Omega \text{ such that } \tau(\omega) < \infty, \\ 0 & \text{for } \tau(\omega) = \infty. \end{cases}$$

It now follows that

$$\lim_{n\to\infty} \mathbf{Q}^n\varphi_{u^*}(x, s) = \varphi_{\hat{u}}(x, s) \quad \text{for } (x, s) \in \Sigma \times R^+.$$

From (4.7) it follows that

$$\varphi_{u^*}(x', s') > \mathbf{Q}\varphi_{u^*}(x', s').$$

Since $f \leqq g$ implies $\mathbf{Q}f \leqq \mathbf{Q}g$, we have

$$\varphi_{u^*}(x', s') > \mathbf{Q}\varphi_{u^*}(x', s') \geqq \lim_{n\to\infty} \mathbf{Q}^n\varphi_{u^*}(x', s') = \varphi_{\hat{u}}(x', s')$$

which contradicts the optimality of u^* in U'. Hence $\varphi_{u^*}(x, s) \leqq \psi(x, s, U')$ for $(x, s) \in \Sigma \times R^+$, and the theorem is proved.

Let $\hat{\psi}$ be the function obtained by setting $\lambda = 0$ in the definition ψ. Then one may prove the following corollary which gives a necessary condition for an optimal control in the undiscounted case.

COROLLARY 4.4. *Let the conditions of Theorem* 4.3 *be satisfied. In addition, assume that* $\tau(\omega) < \infty$ *for a.e.* $\omega \in \Omega$. *If* $u^* \in U'$ *and*

$$\hat{\varphi}_{u^*}(z) = \inf_{u\in U'} \hat{\varphi}(z) \quad \textit{for } z \in \Sigma \times R^+,$$

then

$$\hat{\varphi}_{u^*}(z) = \hat{\psi}(z, U') \quad \textit{for } z \in \Sigma \times R^+.$$

Proof. Since $\tau(\omega) < \infty$ for a.e. $\omega \in \Omega$, one may use the same method of proof as given for Theorem 4.3 with $\lambda = 0$. This proves the corollary.

For $u = (\gamma, \beta) \in U$, define the operator $\mathbf{V}^u$ on functions $f \in F$ such that $f^+(x, \cdot)$ exists for $x \in \Sigma$ as follows:

$$\mathbf{V}^u f(x, s) = f^+(x, s) + \gamma(x, s)\left[\int_\Sigma f(y, 0)k(x, \beta(x, s), dy) - f(x, s)\right]$$

for $(x, s) \in \Sigma \times R^+$. Note that $\mathbf{A}^u$ given in Lemma 3.1 is a restriction of $\mathbf{V}^u$.

The following theorem gives the main result of this paper. This theorem shows that under the stated conditions, the dynamic programming conditions for control of a stochastic process are necessary and sufficient for the optimality of u^*. An analogue of this theorem for undiscounted cost is given in the corollary below.

THEOREM 4.5. *Let* $U' \subset U$ *be the set of controls* $u = (\gamma, \beta)$ *such that* $\gamma \leqq M \in R^+$ *and for each* $x \in \Sigma$,

(i) $\gamma(x, \cdot)$ *is right continuous,*

(ii) $\beta(x, \cdot)$ *is piecewise constant, right continuous, and has only a finite number of discontinuities.*

Suppose that:

(iii) *C and c are bounded,*

(iv) *$c(x, \cdot, u(x, \cdot))$ is right continuous for $u \in U'$ and $x \in \Sigma$,*

(v) *for $x \in \pi(\Delta)$, $C^+(x, \cdot)$ exists, is bounded, and is right continuous.*

Then u^* is optimal in U' if and only if $u^* \in U'$ and

$$\lambda\varphi_{u^*}(z) = \min_{u \in U'} \{c(z, u(z)) + \mathrm{A}^u \varphi_{u^*}(z)\} \quad \text{for } z \in \Sigma \times R^+ - \Delta. \tag{4.9}$$

Proof. Suppose u^* is optimal in U' and that $(x, s) \in \Sigma \times R^+ - \Delta$. Since $\{s : (x, s) \in \Delta\}$ is closed, we may choose h so small that

$$\sigma_1 = \min\{h, \delta_1\}.$$

Observe that U' is closed under one point exchanges and that since $\gamma \leqq M$, $\zeta(\omega) = \infty$ for a.e. $\omega \in \Omega$. One may check that the remaining hypotheses of Theorem 4.3 are satisfied. Let $u = (\gamma, \beta) \in U'$. By Theorem 4.3, we have for $(x, s) \in \Sigma \times R^+ - \Delta$,

$$\begin{aligned}\varphi_{u^*}(x, s) &\leqq W(x, s, u) + \int e^{-\lambda r}\varphi_{u^*}(z) H_u(x, s, dz, dr) \\ &= \frac{1 - a_\gamma(x, s + h)}{1 - a_\gamma(x, s)}\left[\int_0^h e^{-\lambda r} c(x, s + r, u(x, s + r))\, dr + e^{-\lambda h}\varphi_{u^*}(x, s + h)\right] \\ &\quad + \int_0^h \frac{a_\gamma^+(x, s + r)}{1 - a_\gamma(x, s)}\left[\int_0^r e^{-\lambda w} c(x, s + w, u(x, s + w))\, dw \right. \\ &\quad \left. + e^{-\lambda r}\int_\Sigma \varphi_{u^*}(y, 0) k(x, \beta(x, s + r), dy)\right] dr.\end{aligned}$$

Hence, for positive h small enough,

$$\begin{aligned}&\frac{\varphi_{u^*}(x, s) - e^{-\lambda h}\varphi_{u^*}(x, s + h)}{h} \\ &\leqq \frac{1 - a_\gamma(x, s + h)}{1 - a_\gamma(x, s)} \frac{1}{h}\int_0^h e^{-\lambda r} c(x, s + r, u(x, s + r))\, dr \\ &\quad + \frac{1}{h}\int_0^h \frac{a_\gamma^+(x, s + r)}{1 - a_\gamma(x, s)}\left[\int_0^r e^{-\lambda w} c(x, s + w, u(x, s + w))\, dw \right. \\ &\quad \left. + e^{-\lambda r}\int_\Sigma \varphi_{u^*}(y, 0) k(x, \beta(x, s + r), dy) - e^{-\lambda h}\varphi_{u^*}(x, s + h)\right] dr.\end{aligned} \tag{4.10}$$

Since c and C are bounded, so is φ_{u^*}. This combined with the boundedness of γ gives that the right-hand side of (4.10) is bounded by a positive number, say K. Then for h small enough, $\varphi_{u^*}(x, s) - e^{-\lambda h}\varphi_{u^*}(x, s + h) \leqq hK$, and

$$\lim_{h \to 0+} e^{-\lambda h}\varphi_{u^*}(x, s + h) = \varphi_{u^*}(x, s).$$

Note that assumption (ii) of the theorem guarantees that for h small enough,

$\beta(x, s + r, \cdot) = \beta(x, s, \cdot)$ for $0 \leqq r \leqq h$ so that

$$\lim_{r \downarrow 0} \int_\Sigma \varphi_{u^*}(y, 0)k(x, \beta(x, s + r), dy) = \int_\Sigma \varphi_{u^*}(y, 0)k(x, \beta(x, s), dy).$$

If $\varphi_{u^*}^+$ exists, then by taking the limit as $h \to 0+$ in (4.10), we obtain

$$\begin{aligned} &-\varphi_{u^*}^+(x, s) + \lambda\varphi_{u^*}(x, s) \\ (4.11)\qquad &\leqq c(x, s, u(x, s)) + \gamma(x, s)\left[\int_\Sigma \varphi_{u^*}(y, 0)k(x, \beta(x, s), dy) - \varphi_{u^*}(x, s)\right]. \end{aligned}$$

Since the u chosen above is an arbitrary member of U', we have

$$(4.12)\qquad \lambda\varphi_{u^*}(x, s) \leqq \inf_{u \in U'} \{c(x, s, u(x, s)) + \mathbf{V}^u\varphi_{u^*}(x, s)\} \quad \text{for } (x, s) \in \Sigma \times R^+ - \Delta$$

provided $\varphi_{u^*}^+$ exists.

We have already shown that $\varphi_{u^*}(x, \cdot)$ is right continuous at (x, s) for $(x, s) \in \Sigma \times R^+ - \Delta$. Thus, from (4.11) (taking $u = u^*$ and observing that equality holds), we obtain that $\varphi_{u^*}^+(x, s)$ exists and is bounded for $(x, s) \in \Sigma \times R^+ - \Delta$ and furthermore that $\varphi_{u^*}^+(x, \cdot)$ is right continuous at s. Since $\varphi_{u^*}(x, s) = C(x, s)$ for $(x, s) \in \Delta$, we have that φ_{u^*} satisfies the conditions on f in Lemma 3.1. Thus, for $u \in U'$, Lemma 3.1 yields that φ_{u^*} is in the domain of $\mathbf{A}^u$ and that $\mathbf{A}^u\varphi_{u^*} = \mathbf{V}^u\varphi_{u^*}$. Making this substitution in (4.12) we find that (4.9) is necessary for u^* to be optimal in U'.

Suppose $u^* \in U'$ and that φ_{u^*} satisfies (4.9). Clearly, $\varphi_{u^*}(z) = C(z)$ for $z \in \Delta$. By the preceding paragraph φ_{u^*} is in the domain of $\mathbf{A}^u$ for $u \in U'$. Thus Theorem 4.1 yields that u^* is optimal in U'. This proves the theorem.

COROLLARY 4.6. *Let U' be a class of controls u which satisfy the conditions* (i)–(v) *of Theorem* 4.5 *and for which*

$$(4.13)\qquad E_z[\tau] < \infty \quad \textit{for } z \in \Sigma \times R^+.$$

Suppose $u^* \in U'$. Then

$$\hat{\varphi}_{u^*}(z) = \inf_{u \in U'} \hat{\varphi}_u(z) \quad \textit{for } z \in \Sigma \times R^+$$

if and only if

$$(4.14)\qquad 0 = \min_{u \in U'} \{c(z, u(z)) + \mathbf{A}^u\hat{\varphi}_{u^*}(z)\} \quad \textit{for } z \in \Sigma \times R^+ - \Delta.$$

Proof. Since (4.13) holds, we have $\tau(\omega) < \infty$ for a.e. $\omega \in \Omega$. Thus we may follow the proof of Theorem 4.5 with $\lambda = 0$ and apply Corollary 4.4 to show that (4.14) is necessary for the optimality of u^*. The boundedness of C and c combined with (4.13) imply that $\hat{\varphi}_{u^*}$ is bounded. An argument essentially identical to that given in the proof of Theorem 4.1 shows that $\hat{\varphi}_{u^*}$ is in the domain of $\mathbf{A}^u$ for $u \in U'$. The sufficiency of (4.14) then follows from Corollary 4.2.

Remark 4.7. Let $\tilde{U}$ be the class of controls $u = (\gamma, \beta)$ such that

$$\gamma : \Sigma \to R^+, \qquad \beta : \Sigma \to B,$$

where γ and β are Borel measurable, and $k(\cdot, \beta, \Gamma)$ is Borel measurable for each $\Gamma \in \mathscr{S}_1$.

In this case the jump time distributions for $\{X_t^u\}$ become negative exponential and the after jump distributions, $k(x, \beta(x), \cdot)$, do not depend on the length of time the process $\{X_t^u\}$ spends in state x. Thus the process is Markovian.

Suppose, in addition, that $\Delta = \Sigma' \times [0, \infty]$ for some $\Sigma' \subset \Sigma$ and that for $u \in \tilde{U}$,

$$\varphi_u(x) = E_x\left[\int_0^\tau e^{-\lambda t}\tilde{c}(X_t, u(X_t))\,dt + e^{-\lambda\tau}\tilde{C}(X_\tau)\right] \quad \text{for } x \in \Sigma,$$

$$\hat{\varphi}_u(x) = E_x\left[\int_0^\tau \tilde{c}(X_t, u(X_t))\,dt + \tilde{C}(X_\tau)\right] \quad \text{for } x \in \Sigma,$$

where $\tilde{c}$ and $\tilde{C}$ are bounded Borel functions. For $u \in \tilde{U}$, let $\tilde{\mathrm{A}}^u$ be the weak infinitesimal operator of $\{X_t^u\}$. Then

$$\tilde{\mathrm{A}}^u f(x) = \gamma(x)\left[\int_\Sigma f(y)k(x, \beta(x), dy) - f(x)\right]$$

for any bounded Borel measurable function $f: \Sigma \to R$. A straightforward modification of the proof of Theorem 4.5 shows that if γ is bounded, u^* is optimal in $\tilde{U}$ if and only if

$$\lambda\varphi_{u^*}(x) = \inf_{u \in \tilde{U}}\{\tilde{c}(x, u(x)) + \tilde{\mathrm{A}}^u\varphi_{u^*}(x)\} \quad \text{for } x \in \Sigma. \tag{4.15}$$

For the above situation, Kushner [12, p. 527] has observed that (4.15) is sufficient for optimality.

Similarly, for the undiscounted case, one may show that $\gamma < M \in R^+$ and $E[\tau] < \infty$ implies that u^* is optimal in $\tilde{U}$ if and only if

$$0 = \inf_{u \in \tilde{U}}\{\tilde{c}(x, u(x)) + \tilde{\mathrm{A}}^u\hat{\varphi}_{u^*}(x)\} \quad \text{for } x \in \Sigma.$$

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