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NECESSARY AND SUFFICIENT CONDITIONS FOR OPTIMAL SOLUTIONS TO A SURVIVOR SEARCH PROBLEM*

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The survivor search problem is to find a search plan that maximizes the probability of finding a stationary target alive when the target's position is given by a probability distribution on a Borel subset X of Euclidean n-space and the target's lifetime is a random variable whose distribution depends on the location of the target.

Mathematically, the problem is defined in terms of a prior density function p for the target's location; a lifetime function α, such that $\alpha(x, t)$ is the probability that the target's lifetime is less than or equal to t given the target is located at $x \in X$; and a detection function b such that $b(x, z)$ gives the probability of detecting the target with z effort density applied to the point x given the target is located at x.

A search plan $\psi : X \times [0,\infty) \to [0,\infty)$ specifies $\psi(x, t)$ the rate at which effort density accumulates at point x at time t. Let $\alpha'(x, \cdot)$ and $b'(x, \cdot)$ denote the derivative of $\alpha(x, \cdot)$ and $b(x, \cdot)$. If $b(x, \cdot)$ is concave and $|b'(x, \cdot)| \le \kappa$ for $x \in X$, then a necessary and sufficient condition for a bounded plan ψ^* to maximize the probability of detecting the target alive among all bounded plans which satisfy $\int_X \psi(x, t)\mathrm{d}x = m_2(t)$ for $t \ge 0$ is the existence of a function $\lambda = [0,\infty) \to [0,\infty)$ such that

$$p(x) \int_s^\infty \alpha'(x, t) b'\left(x, \int_0^t \psi(x, u)\mathrm{d}u\right) \mathrm{d}t = \lambda(s) \quad \text{for} \quad \psi^*(x, s) > 0,$$

$$\le \lambda(s) \quad \text{for} \quad \psi^*(x, s) = 0.$$

1. Introduction

We consider the problem of finding a stationary target alive when the target's lifetime is a random variable whose distribution depends on the target's location. This type of search could occur when looking for a wrecked aircraft. Generally, survivors of an airplane crash remain in the vicinity of

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the aircraft so that the target or object of the search is stationary. If the aircraft crashed in an area where food, water, and shelter are available by means of foraging, the passengers might have a longer expected survival time than if the crash occurred in a barren area where these basic necessities are not readily available. If the aircraft filed a flight plan before departing, one could develop a subjective prior probability distribution for the target's location based on the flight plan and any knowledge about the crash location.

Suppose that there is a constraint on the rate at which search effort can be applied. This is usually the case when there are a fixed number of search vehicles, say planes, available for use in the search. In the above situation, the prior probability that the target is located in the first area may be substantially larger than the prior probability that the target is in the barren area. However, if the target is in the barren area, it has a much shorter expected lifetime. The problem in this case is to allocate the available search effort between the two areas to maximize the probability of detecting the target alive.

In Section 2 we state the survivor lifetime problem in a mathematical form, and in Section 3 we find necessary and sufficient conditions for a plan to maximize the probability of finding a target alive. Section 4 presents the optimal plan for a two-celled target distribution when the detection function and lifetime function are both exponential. This plan is obtained by making a guess for the optimal plan and verifying that it satisfies the sufficient conditions of Theorem 3.2. In order to make the results of this paper more useful in operations, one needs to be able to solve the problem of finding optimal plans for target distributions with any number of cells and for more realistic lifetime functions than the exponential one. At present, it is not clear how difficult it is to use the necessary and sufficient conditions of Theorem 3.2 to find optimal plans for this case.

2. Statement of problem

The searcher is looking for a stationary target located in a Borel subset X of Euclidean n-space whose lifetime is random. The object is to detect the target alive. Mathematically, the problem is defined in terms of the following three functions:

(i) $p : X \to [0, \infty)$, the prior probability density for the target's location;

(ii) $\alpha : X \times [0, \infty) \to [0, 1]$, the lifetime function, i.e., $\alpha(x, t)$ is the probability that the target's lifetime is less than or equal to t, given that the target is located at x; and

(iii) $b : X \times [0,\infty) \to [0,1]$, the detection function, i.e., $b(x,z)$ is the probability of detecting the target with z effort density applied to the point x given the target is located at x. We assume $b(x,0)=0$ for $x \in X$.

The functions p, α, and b are assumed to be Borel measurable. A *search plan density* is a Borel function $\psi : X \times [0,\infty) \to [0,\infty)$. Following Stone and Richardson [6], we specify constraints on the amount of effort or rate at which effort may be applied by functions $m_1 : X \times [0,\infty) \to [0,\infty)$, $m_2 : [0,\infty) \to [0,\infty)$, and $m_3 : [0,\infty) \to [0,\infty)$. We require that

$$\begin{aligned}
&\text{(a)} \qquad \psi(x,t) \le m_1(x,t) \quad \text{for} \quad x \in X, \quad t \ge 0, \\
&\text{(b)} \qquad \int_X \psi(x,t)\,\mathrm{d}x \le m_2(t) \qquad \text{for} \quad t \ge 0, \\
&\text{(c)} \qquad \int_X \int_0^t \psi(x,s)\,\mathrm{d}s\,\mathrm{d}x \le m_3(t) \qquad \text{for} \quad t \ge 0.
\end{aligned} \tag{2.1}$$

The function ψ specifies $\psi(x,t)$, the rate at which effort density is accumulating at the point x at time t. Intuitively, condition (a) limits the rate at which search density may be applied to small areas. Condition (b) constrains the rate at which effort may be applied to the whole search space, and (c) limits the total amount of effort available by time t. The class of search plan densities ψ which satisfy conditions (a), (b) and (c) is denoted by $\Psi(m_1, m_2, m_3)$. If the constraint corresponding to m_j is not imposed, we set $m_j = \infty$. For convenience of notation, we define

$$\Psi_2(m_2) = \Psi(\infty, m_2, \infty).$$

Assume that $\alpha'(x,\cdot)$, the derivative of $\alpha(x,\cdot)$ exists on the positive half line and that

$$\alpha(x,t) - \alpha(x,s) = \int_s^t \alpha'(x,u)\,\mathrm{d}u \quad \text{for} \quad t > s > 0. \tag{2.2}$$

That is, we allow an atom at zero (i.e., the target may have died before the search begins) but require the lifetime distribution to be absolutely continuous on the positive half line. If one uses the plan ψ and the target is located at x, then the probability of detecting the target by time t is $b(x, \int_0^t \psi(x,s)\,\mathrm{d}s)$ and the probability $\mathbf{P}_T[\psi \mid x]$ of detecting the target alive by time T when using plan ψ is given by

$$\mathbf{P}_T[\psi \mid x] = \int_0^T \alpha'(x,t)\, b\left(x, \int_0^t \psi(x,s)\,\mathrm{d}s\right) \mathrm{d}t + (1-\alpha(x,T))\, b\left(x, \int_0^T \psi(x,s)\,\mathrm{d}s\right). \tag{2.3}$$

The unconditional probability $\mathbf{P}_T[\psi]$ of detecting the target alive by time T is computed by $\mathbf{P}_T[\psi] = \int_X \mathbf{P}_T[\psi \mid x]\, p(x)\,\mathrm{d}x$.

In [1], conditions are given under which one may find a plan $\psi^* \in \Psi(m_1, \infty, m_3(T))$ such that

$$\mathbf{P}_T[\psi^*] = \max\{\mathbf{P}_T[\psi]:\ \psi \in \Psi(m_1, \infty, m_3(T))\}.$$

Examples of such plans are also given.

In this paper we consider $\mathbf{P}_\infty[\psi]$, the probability of detecting the target alive using plan ψ when there is no time limit on the search. There is, however, a constraint $m_2(t)$ on the rate at which search effort may be applied at time t for $t \geq 0$. Specifically, we seek a plan $\psi^* \in \Psi_2(m_2)$ such that

$$\mathbf{P}_\infty[\psi^*] = \max\{\mathbf{P}_\infty[\psi]:\ \psi \in \Psi_2(m_2)\}.$$

Such a plan will be called an *optimal survivor search plan* within $\Psi_2(m_2)$.

If $\lim_{t\to\infty} \alpha(x,t) = 1$ for $x \in X$ (i.e., with probability one, the target has a finite lifetime), then (2.3) becomes

$$\mathbf{P}_\infty[\psi \mid x] = \int_0^\infty \alpha'(x,t)\, b\left(x, \int_0^t \psi(x,s)\,\mathrm{d}s\right) \mathrm{d}t$$

and

$$\begin{aligned}\mathbf{P}_\infty[\psi] &= \int_X p(x) \int_0^\infty \alpha'(x,t)\, b\left(x, \int_0^t \psi(x,s)\,\mathrm{d}s\right) \mathrm{d}t\,\mathrm{d}x \\ &= \int_0^\infty \int_X p(x)\alpha'(x,t)\, b\left(x, \int_0^t \psi(x,s)\,\mathrm{d}s\right) \mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{2.4}$$

Let $m_3(t) = \int_0^t m_2(s)\,\mathrm{d}s$ for $t \geq 0$. If one can find $\psi^* \in \Psi(\infty, \infty, m_3)$ such that for each $t \geq 0$, ψ^* maximizes

$$\int_X p(x)\,\alpha'(x,t)\, b\left(x, \int_0^t \psi(x,s)\,\mathrm{d}s\right) \mathrm{d}x \tag{2.5}$$

subject to $\psi \in \Psi(\infty, \infty, m_3)$, then it is clear from equation (2.4) that ψ^* satisfies

$$\mathbf{P}_\infty[\psi^*] = \max\{\mathbf{P}_\infty[\psi]:\ \psi \in \Psi(\infty, \infty, m_3)\}.$$

Consider for a moment the situation where the target's lifetime distribution does not depend on x. Then maximizing (2.5) is equivalent to maximizing

$$\int_X p(x)\, b\left(x, \int_0^t \psi(x,s)\,\mathrm{d}s\right) \mathrm{d}x$$

subject to

$$\int_X \int_0^t \psi(x,s)\,\mathrm{d}s\,\mathrm{d}x \leq m_3(t) \quad \text{for} \quad t \geq 0.$$

By letting $\varphi(x,t) = \int_0^t \psi(x,s)\mathrm{d}s$, one can check that the above problem is solved by finding a uniformly optimal plan φ^* for a stationary target with prior probability distribution given by p. (See [5] for definition of uniformly optimal plans.) In fact, Theorems 2.2.4 and 2.4.6 of [5] give methods of finding uniformly optimal plans. Thus, the interesting survivor lifetime problem arises when α varies with x.

Returning to (2.5), we could, for each t, find a function $f_t : X \to [0, \infty)$ such that f_t maximizes

$$\int_X p(x)\, \alpha'(x,t)\, b(x, f_t(x))\mathrm{d}x$$

subject to

$$\int_X f_t(x)\mathrm{d}x \leq m_3(t).$$

If $\int_X p(x)\alpha'(x,t)\mathrm{d}x < \infty$ and if $b(x,0) = 0$, $b'(x,\cdot)$ is positive, continuous, and strictly decreasing for $x \in X$ (i.e., b is regular), then one may compute f_t by the method given in Theorem 2.4.3 of [5]. Let $\varphi(\cdot, t) = f_t$ for $t \geq 0$. If the result of solving the above problem for each $t \geq 0$ is a collection of functions $\{f_t : t \geq 0\}$ such that $\varphi'(x,\cdot)$, the derivative of $\varphi(x,\cdot)$, exists for $x \in X$, and $\varphi'(x,t) \geq 0$ for $x \in X$ and $t \geq 0$, then $\psi^* = \varphi'$ is an optimal survivor lifetime plan. However, there are many situations of interest for which the function φ obtained in this manner is not increasing. Case 1 of the example in Section 4 presents such a situation. Thus, the solution to the survivor search problem requires methods beyond the standard ones already developed for search problems. Such methods are developed in the next section.

3. Necessary and sufficient conditions for optimal survivor lifetime plans

In order to find necessary conditions for optimal survivor lifetime plans, we use Theorem 6.1 in [3] which is reproduced as Theorem 3.1 below. For the convenience of the reader, some of the terminology used in this theorem is defined below.

Theorem 3.1 gives necessary conditions for the achievement of a constrained minimum of a functional F defined on a locally convex topological vector space E. The constraints are expressed by specifying subsets $Q_i \subset E$ for $i = 1, \ldots, k+1$ and requiring that $f^* \in Q = \bigcap_{i=1}^{k+1} Q_i$ be such that

$$F(f^*) = \min\{F(f): f \in Q\}.$$

The sets Q_i, $i = 1, \ldots, k$, all must contain interior points while Q_{k+1} must have none. Typically, the sets Q_i, $i = 1, \ldots, k$, represent inequality constraints while Q_{k+1} represents an equality constraint. For the purpose of retaining motivation, the sets Q_i, $i = 1, \ldots, k$, are called *inequality sets* while Q_{k+1} is called an *equality set*. Note that there is only one equality set.

A set K in a vector space is called a *cone with apex at* 0 if $h \in K$ implies $\lambda h \in K$ for all $\lambda > 0$. Let $\bar{E}$ be the set of continuous linear functionals defined on E. Let $K \subset E$ be a cone with apex at 0. The dual cone $\bar{K}$ is defined by

$$\bar{K} = \{g \in \bar{E} \mid g(h) \geq 0 \text{ for } h \in K\}.$$

The vector $h \in E$ is a direction of decrease of F at $f_0 \in E$ if there exists a neighborhood U_h of h and a number $\alpha < 0$ such that for some $\varepsilon_0 > 0$

$$F(f_0 + \varepsilon \tilde{h}) \leq F(f_0) + \varepsilon\alpha \quad \text{for} \quad 0 < \varepsilon < \varepsilon_0 \quad \text{and} \quad \tilde{h} \in U_h.$$

One can show that the directions of decrease generate an open cone with apex at 0. The functional F is *regularly decreasing* at $f_0 \in E$ if the set of its directions of decrease at f_0 is convex.

Let $Q \subset E$. The vector $h \in E$ is a *feasible direction for Q at $f_0 \in E$* if there exists a neighborhood U_h of h such that for some $\varepsilon_0 > 0$

$$f_0 + \varepsilon \tilde{h} \in Q \quad \text{for} \quad 0 < \varepsilon < \varepsilon_0 \quad \text{and} \quad \tilde{h} \in U_h.$$

The set of feasible directions for Q at f_0 forms an open cone with apex at 0. The set Q is *feasible regular* at $f_0 \in E$ if the cone of feasible directions for Q at f_0 is convex.

The vector $h \in E$ is a *tangent direction to Q at $f_0 \in E$* if there exists $\varepsilon_0 > 0$ and a function $q: (0, \varepsilon_0) \to Q$ such that if

$$r(\varepsilon) = q(\varepsilon) - (f_0 + \varepsilon h),$$

then for any neighborhood U of the zero vector, there exists $0 < \delta < \varepsilon_0$ such that

$$\frac{1}{\varepsilon} r(\varepsilon) \in U \quad \text{for} \quad 0 < \varepsilon < \delta.$$

In a Banach space this last condition may be replaced by $\|r(\varepsilon)\| = o(\varepsilon)$. One may verify that the tangent directions to Q at f_0 form a cone with apex at 0, but this cone is, in general, neither open nor closed. Note that every feasible direction is also a tangent direction but the converse is not true. The set $Q \subset E$ is *tangent regular* at f_0 if the cone of tangent directions to Q at f_0 is convex.

We now state the theorem of Dubovitskii and Milyutin which we will use in proving our results.

Theorem 3.1 (Dubovitskii–Milyutin). *Let the functional F be defined on a locally convex topological vector space E. Let $Q_1, \ldots, Q_k$ be inequality subsets of E and Q_{k+1} be an equality subset of E. Let F assume a minimum at $f^* \in Q = \bigcap_{i=1}^{k+1} Q_i$. Assume* (1) *$F$ is regularly decreasing at f^* with cone of directions of decrease K_0*; (2) *for $i = 1, \ldots, k$, Q_i is feasible regular at f^* with cone of feasible directions K_i which is non-empty*; *and* (3) *the set Q_{k+1} is tangent regular at f^* with a cone of tangent directions K_{k+1} which contains at least one non-zero vector. Then there exist continuous linear functionals g_i, $i = 0, \ldots, k+1$, not all identically zero, such that $g_i \in \bar{K}_i$ = dual cone of K_i for $i = 0, \ldots, k+1$ which satisfy the Euler–Lagrange equation*

$$g_0 + g_1 + \ldots + g_{k+1} = 0. \tag{3.1}$$

Let L_∞ be the Banach space of real-valued, bounded, measurable functions f defined on $X \times [0, \infty)$ with norm

$$\|f\| = \operatorname*{ess\,sup}_{(x,t) \in X \times [0,\infty)} |f(x,t)|.$$

The measure on $X \times [0, \infty)$ is assumed to be $(n+1)$-dimensional Lebesgue measure. The following is the main theorem of this paper.

Theorem 3.2. *For $x \in X$, suppose that $\alpha(x, \cdot)$ is absolutely continuous on $(0, \infty)$, $\int_0^\infty t\alpha'(x,t)\,dt \le \tau < \infty$, and that $b'(x, \cdot)$, the derivative of $b(x, \cdot)$ satisfies $0 \le b'(x,z) \le \kappa < \infty$ for $z \ge 0$. Then* (a) *and* (b) *below are true.*

(a) *Suppose that X is compact and $b'(x,z) > 0$ for $x \in X$ and $z \ge 0$. If ψ^* is a bounded optimal survival search plan within $\Psi_2(m_2)$ and m_2 is a positive bounded Borel function, then there exists $\lambda : [0,\infty) \to [0,\infty)$ such that for a.e. $(x,s) \in X \times [0,\infty)$,*

$$p(x)\int_s^\infty \alpha'(x,t)\,b'\left(x,\int_0^t \psi^*(x,u)\mathrm{d}u\right)\mathrm{d}t = \lambda(s) \quad \text{for} \quad \psi^*(x,s)>0$$
$$\le \lambda(s) \quad \text{for} \quad \psi^*(x,s)=0. \qquad (3.2)$$

(b) *Suppose $b(x,\cdot)$ is concave for $x\in X$. If there is a bounded $\psi^*\in\Psi_2(m_2)$ such that*

$$\int_X \psi^*(x,s)\mathrm{d}x = m_2(s) \quad \textit{for} \quad \textit{a.e.} \quad s\in[0,\infty)$$

and a Borel function $\lambda:[0,\infty)\to[0,\infty)$ such that ψ^* and λ satisfy (3.2), *then ψ^* is an optimal survivor search plan among bounded plans in $\Psi_2(m_2)$.*

Proof. First we prove part (a). In order to use Theorem 3.1, we extend the definition of b so that

$$b(x,z)=b(x,0)+zb'(x,0) \quad \text{for} \quad z<0, \quad x\in X.$$

Then in the terminology of Theorem 3.1, we let

$$F(f) = -\mathbf{P}_\infty[f] \quad \text{for} \quad f\in L_\infty,$$
$$E = L_\infty,$$
$$Q_1 = L_\infty\cap\{f: f\ge 0\},$$
$$Q_2 = L_\infty\cap\left\{f:\int_X f(x,t)\mathrm{d}x = m_2(t) \quad \text{for} \quad \text{a.e.} \quad t\ge 0\right\}.$$

Note that Q_1 has interior points and that Q_2 has none in the L_∞ topology.

Suppose that ψ^* is a bounded optimal survivor lifetime plan within $\Psi_2(m_2)$. Since $b'(x,z)>0$ for $x\in X$ and $z\ge 0$, it is clear that ψ^* must be a member of Q_2 and that the optimization may be restricted to functions in Q_2.

We find K_0 by the use of Theorem 7.3 in [3]. To do this, we first show that F satisfies a Lipschitz condition. For $f_1, f_2\in L_\infty$,

$$|F(f_2)-F(f_1)| \le \int_X\int_0^\infty p(x)\alpha'(x,t)\left|b\left(x,\int_0^t f_1(x,s)\mathrm{d}s\right) - b\left(x,\int_0^t f_2(x,s)\mathrm{d}s\right)\right|\mathrm{d}t\,\mathrm{d}x$$
$$\le \int_X\int_0^\infty p(x)\alpha'(x,t)t\kappa\,\|f_1-f_2\|\,\mathrm{d}t\,\mathrm{d}x$$
$$\le \tau\kappa\,\|f_1-f_2\|.$$

Let $F'(\psi^*, h)$ be the derivative of F at ψ^* in the direction of h, that is,

$$F'(\psi^*, h) = \lim_{\varepsilon \to 0+} \frac{1}{\varepsilon} [F(\psi^* + \varepsilon h) - F(\psi^*)] \quad \text{for} \quad h \in L_\infty$$

whenever the limit exists. Thus,

$$F'(\psi^*, h) = \lim_{\varepsilon \to 0+} \int_X \int_0^\infty p(x)\alpha'(x, t)\frac{1}{\varepsilon}\left\{b\left(x, \int_0^t \psi^*(x, s)\mathrm{d}s\right) - b\left(x, \int_0^t [\psi^*(x, s) + \varepsilon h(x, s)]\mathrm{d}s\right)\right\} \mathrm{d}t\, \mathrm{d}x.$$

Since the integrand is dominated by $p(x)\alpha'(x, t)t\kappa \|h\|$, we may apply the dominated convergence theorem and then interchange orders of integration to obtain

$$\begin{aligned} F'(\psi^*, h) &= -\int_X \int_0^\infty p(x)\alpha'(x, t)\, b'\left(x, \int_0^t \psi^*(x, u)\mathrm{d}u\right) \int_0^t h(x, s)\mathrm{d}s\, \mathrm{d}t\, \mathrm{d}x \\ &= -\int_X \int_0^\infty p(x)\left[\int_s^\infty \alpha'(x, t)\, b'\left(x, \int_0^t \psi^*(x, u)\mathrm{d}u\right) \mathrm{d}t\right] h(x, s)\mathrm{d}s\, \mathrm{d}x. \end{aligned} \tag{3.3}$$

Thus, $F'(\psi^*, \cdot)$ is a bounded linear operator on L_∞, and by Theorem 7.3 in [3], F is regularly decreasing at ψ^* and the cone of directions of decrease at ψ^* is

$$K_0 = \{h: F'(\psi^*, h) < 0\}. \tag{3.4}$$

Let $\bar{L}_\infty$ denote the set of continuous linear functionals on L_∞. By Theorem 10.2 in [3],

$$\bar{K}_0 = \{g \in \bar{L}_\infty\colon g = -\delta F'(\psi^*, \cdot) \text{ for some } 0 \le \delta < \infty\}. \tag{3.5}$$

Let Q_1^0 be the interior of Q_1. The cone of feasible directions for Q_1 at ψ^* is obtained from Theorem 8.2 in [3],

$$K_1 = \{h \in L_\infty\colon h = \delta(f - \psi^*) \text{ for some } f \in Q_1^0,\ \delta > 0\}.$$

Since K_1 is convex, Q_1 is regular at ψ^*. From Theorem 10.5 in [3] we obtain

$$\bar{K}_1 = \{g \in \bar{L}_\infty\colon g(f) \ge g(\psi^*) \text{ for all } f \in Q_1\}. \tag{3.6}$$

By Theorem IV.8.16 in [2], the set $\bar{L}_\infty$ is isomorphically isometric to B, the set of *finitely* additive bounded set functions defined on $\mathcal{M}$, the Lebesgue measurable subsets of $X \times [0, \infty)$. Moreover, each $\beta \in B$ corresponds, under this isomorphism, to a $g \in \bar{L}_\infty$ in the following manner:

$$g(h)=\int_{X\times[0,\infty)} h\,\mathrm{d}\beta \quad \text{for} \quad h\in L_\infty. \tag{3.7}$$

Let $g_1\in \bar{K}_1$ and β_1 be the measure corresponding to g_1. Then $\beta_1\geq 0$. To see this, suppose $\beta_1(S)<0$ for some $S\in\mathcal{M}$. Let

$$f(x,s)=\begin{cases}1+\psi^*(x,s) & \text{for} \quad (x,s)\in S,\\ \psi^*(x,s) & \text{for} \quad (x,s)\in X\times[0,\infty)-S.\end{cases}$$

Then $g_1(f-\psi^*)=\beta_1(S)<0$ in contradiction to (3.6). Let $f_0(x,s)=0$ for $(x,s)\in X\times[0,\infty)$. Then $f_0\in Q_1$ and $0=g_1(f_0)\geq g_1(\psi^*)$, the inequality following from (3.6). Since $\beta_1\geq 0$, it follows that $g_1(\psi^*)=0$ and $\beta_1(S)=0$ for $S\subset R=\{(x,s)\colon \psi^*(x,s)>0\}$.

Thus, $\bar{K}_1$ is isomorphic to

$$\{\beta\in B\colon \beta\geq 0 \text{ and } \beta(R)=0\}. \tag{3.8}$$

In order to find K_2 and $\bar{K}_2$, we define an operator A from L_∞ into the set of bounded Lebesgue measurable functions on $[0,\infty)$ by

$$A(f)=\int_X f(x,\cdot)\,\mathrm{d}x-m_2.$$

Clearly, $A(\psi^*)=0$. Since A is an affine operator, the Frechet differential A' of A is given by

$$A'(f)=\int_X f(x,\cdot)\,\mathrm{d}x.$$

Let l denote Lebesgue measure on X. Since X is compact, it follows that $l(X)<\infty$, A is defined on L_∞, and A' is continuous. Thus, we may use Theorem 9.1 in [3] to obtain

$$K_2=\{h\in L_\infty\colon A'(h)=0\}$$
$$=\{h\in L_\infty\colon \int_X h(x,s)\,\mathrm{d}x=0 \text{ for a.e. } s\in[0,\infty)\}.$$

Since K_2 is a subspace of L_∞, Q_2 is regular at ψ^* and, in addition, we may apply Theorem 10.1 in Girsanov [3] to find that

$$\bar{K}_2=\{g\in\bar{L}_\infty\colon g(h)=0 \text{ for } h\in K_2\}.$$

By Theorem 3.1, then there exist $g_i\in\bar{K}_i$, $i=0,1,2$, not all identically 0 such that

$$g_0+g_1+g_2=0. \tag{3.9}$$

We now show that (3.9) implies (3.2). In order to do this, we show that the measures corresponding to g_1 and g_2 have densities. From (3.3) and (3.5), we already know this is the case for g_0. First we find the density corresponding to g_2.

Claim. There is a finitely additive set-function ν on $[0,\infty)$ such that

$$g_2(h) = \int_0^\infty \int_X h(x,s)\,\mathrm{d}x\,\nu(\mathrm{d}s) \quad \text{for} \quad h \in L_\infty. \tag{3.10}$$

Proof. Let T be a Lebesgue (i.e., Lebesgue measurable) subset of $[0,\infty)$. We shall use I_U for the indicator function of a set U. For Lebesgue $S \subset X$, define

$$\eta(S) = g_2(I_{S\times T}).$$

For the proof, we consider only the case where η is non-negative. The case of a signed measure can be handled by decomposing η into positive and negative measures. Since we are taking η non-negative, we may assume $\eta(X) > 0$.

In order to prove the claim, we show the measure η is regular. Let S_1 be a Lebesgue measurable set of X. Since the Lebesgue measure l is regular, there is for each $\varepsilon > 0$ an open set S_o and closed set S_c (in the relative topology induced on X by Euclidean n-space) such that $S_c \subset S_1 \subset S_o$ and $l(S_o - S_c) < \varepsilon$. Fix $\varepsilon > 0$ and let S_o and S_c be the sets corresponding to ε. Let

$$h_a = I_{(S_o - S_c)\times T}$$

and

$$h_b(x,s) = \begin{cases} \dfrac{l(S_o - S_c)}{l(X)} & \text{for} \quad (x,s) \in X \times T, \\ 0 & \text{otherwise.} \end{cases}$$

Let $h = h_a - h_b$. Then $h \in K_2$ and $g_2(h) = 0$. Observe that

$$\|g_2(h_b)\| \le \|g_2\|\,\|h_b\| = \|g_2\| \frac{l(S_o - S_c)}{l(X)} < \frac{\varepsilon\,\|g_2\|}{l(X)}$$

and

$$\|g_2(h_a)\| \le \|g_2(h)\| + \|g_2(h_b)\| < \frac{\varepsilon\,\|g_2\|}{l(X)}.$$

Thus,

$$|\eta(S_o - S_c)| = |g_2(h_a)| < \frac{\varepsilon \|g_2\|}{l(X)},$$

and η is regular.

Since η is a regular finitely additive measure on the Lebesgue subsets of the compact set X, Theorem III.5.13 in [2] allows us to conclude that η is countably additive. Since $g_2(h) = 0$ for $h \in K_2$, it follows that η is absolutely continuous with respect to Lebesgue measure, and by the Radon–Nikodym theorem, there is a Lebesgue integrable function ξ_T such that

$$\eta(S) = \int_S \xi_T(x)\,\mathrm{d}x \quad \text{for Lebesgue} \quad S \subset X.$$

We now show that ξ_T is a constant function on X. That is, there is a $\nu(T) \geq 0$ such that $\xi_T(x) = \nu(T)$ for a.e. $x \in X$. Suppose this were not the case. Then there would exist sets W and $S \subset X$ such that $l(W) = l(S) > 0$, $W \cap S = \emptyset$, and

$$x_1 \in W, \qquad x_2 \in S \Rightarrow \xi_T(x_1) > \xi_T(x_2).$$

Letting $h = I_{W\times T} - I_{S\times T}$, we have $h \in K_2$ and $g_2(h) = 0$ which contradicts

$$g_2(h) = \eta(W) - \eta(S) = \int_W \xi_T(x)\,\mathrm{d}x - \int_S \xi_T(x)\,\mathrm{d}x > 0.$$

Thus, such a $\nu(T)$ exists for each Lebesgue subset T of $[0, \infty)$ and we may consider ν as a function defined on the Lebesgue subsets of $[0, \infty)$. Since

$$\nu(T) = \frac{g_2(I_{X\times T})}{l(X)},$$

we have that ν is a finitely additive set function. Moreover,

$$g_2(I_{S\times T}) = l(S)\,\nu(T) \text{ for Lebesgue subsets } S \subset X \text{ and } T \subset [0, t].$$

It follows that

$$g_2(h) = \int_0^\infty \int_X h(x, s)\,\mathrm{d}x\,\nu(\mathrm{d}s) \quad \text{for} \quad h \in L_\infty, \tag{3.11}$$

and the claim is proved.

Let

$$D(\psi^*, x, s) = p(x) \int_s^\infty \alpha'(x, t)\, b'\left(x, \int_0^t \psi^*(x, u)\,\mathrm{d}u\right) \mathrm{d}t$$
$$\text{for} \quad (x, s) \in X \times [0, \infty).$$

From (3.3) and (3.5), we obtain, for some $\delta \geq 0$,

$$g_0(h) = \delta \int_X \int_0^\infty D(\psi^*, x, s) h(x, s) \mathrm{d}s \, \mathrm{d}x \quad \text{for} \quad h \in L_\infty . \tag{3.12}$$

Let β_1 be the set function in B corresponding to g_1. Let S be a Lebesgue subset of $R = \{(x, s)\colon \psi^*(x, s) > 0\}$. Let $h = I_S$; then by (3.8), $\beta_1(S) = 0$ and $g_1(h) = 0$. Thus,

$$g_0(h) + g_2(h) = 0,$$

i.e.,

$$\int_S \delta D(\psi^*, x, s) \mathrm{d}s \, \mathrm{d}x = -\int_S \nu(\mathrm{d}s) \mathrm{d}x \quad \text{for Lebesgue} \quad S \subset R. \tag{3.13}$$

Let $R_s = \{x\colon (x, s) \in R\}$ for $0 \leq s < \infty$. Since

$$\int_X \psi^*(x, s) \mathrm{d}x = m_2(s) \quad \text{for} \quad s \in [0, \infty),$$

$l(R_s) > 0$ for a.e. $s \in [0, \infty)$. Thus,

$$\nu([0, u]) = \int_0^u \int_{R_s} \frac{1}{l(R_s)} \mathrm{d}x \, \nu(\mathrm{d}s) \quad \text{for} \quad 0 \leq u < \infty. \tag{3.14}$$

In addition, (3.13) implies the existence of a Lebesgue measurable function $\nu'\colon [0, \infty) \to [0, \infty)$ such that

$$\delta D(\psi^*, x, s) = -\nu'(s) \quad \text{for} \quad (x, s) \in R \tag{3.15}$$

and

$$\int_S \mathrm{d}x \, \nu(\mathrm{d}s) = \int_S \nu'(s) \mathrm{d}x \, \mathrm{d}s \quad \text{for Lebesgue} \quad S \subset R. \tag{3.16}$$

Thus, by (3.14) and (3.16),

$$\nu([0, u]) = \int_0^u \nu'(s) \mathrm{d}s \quad \text{for} \quad 0 \leq s < 0.$$

That, is, ν' is a density for ν. It follows from (3.11) that

$$g_2(h) = \int_0^\infty \int_X h(x, s) \, \nu'(s) \mathrm{d}x \, \mathrm{d}s \quad \text{for} \quad h \in L_\infty .$$

That is, both g_0 and g_2 correspond to measures given by densities so that equation (3.9) yields that the measure β_1 corresponding to g_1 must have a density β_1'. Equation (3.8) then implies

$$\beta_1'(x,s)=0 \quad \text{for} \quad (x,s)\in R,$$

$$\beta_1'(x,s)\geq 0 \quad \text{for} \quad (x,s)\in X\times[0,\infty)-R.$$

Recall from (3.5) that $\delta\geq 0$. It now follows that $\delta>0$. For if $\delta=0$, then by (3.15), $\nu'=0$ which implies by (3.9) that $\beta_1'=0$. This would imply $g_0=g_1=g_2=0$ which contradicts Theorem 3.1. Thus, we may define

$$\lambda(s)=\frac{-\nu'(s)}{\delta} \quad \text{for} \quad 0\leq s<\infty.$$

We then have

$$D(\psi^*,x,s)=\lambda(s) \quad \text{for} \quad (x,s)\in R$$

$$D(\psi^*,x,s)\leq D(\psi^*,x,s)+\beta'(x,s)=\lambda(s) \quad \text{for} \quad (x,s)\in X\times[0,\infty)-R.$$

This proves part (a) of the theorem.

We prove part (b) using an argument by contradiction. Suppose $\psi\in\Psi_2(m_2)$ is bounded and $\mathbf{P}_\infty[\psi]>\mathbf{P}_\infty[\psi^*]$, i.e., $F(\psi)<F(\psi^*)$. Let $h=\psi-\psi^*$. Since $b(x,\cdot)$ is concave for $x\in X$, F is convex. Hence, for $0\leq\varepsilon\leq 1$,

$$\begin{aligned}F(\psi^*+\varepsilon h)-F(\psi^*)&=F((1-\varepsilon)\psi^*+\varepsilon\psi)-F(\psi^*)\\&\leq(1-\varepsilon)F(\psi^*)+\varepsilon F(\psi)-F(\psi^*)\\&=\varepsilon[F(\psi)-F(\psi^*)].\end{aligned}$$

Therefore,

$$F'(\psi^*,h)\leq F(\psi)-F(\psi^*)<0,$$

and

$$\begin{aligned}0<-F'(\psi^*,h)&=\int_0^\infty\int_X D(\psi^*,x,s)[\psi(x,s)-\psi^*(x,s)]\,\mathrm{d}x\,\mathrm{d}s\\&\leq\int_0^\infty\lambda(s)\int_X[\psi(x,s)-\psi^*(x,s)]\,\mathrm{d}x\,\mathrm{d}s \quad \text{(by (3.2))}\\&\leq 0.\end{aligned}$$

The last equality follows from the fact that

$$\int_X\psi(x,s)\,\mathrm{d}x\leq m_2(s)=\int_X\psi^*(x,s)\,\mathrm{d}x \quad \text{for a.e.} \quad s\in[0,\infty).$$

This contradiction completes the proof.

Remark 3.3. The class of bounded functions is not the most natural space to use for search plan densities. Recall the discussion in Section 2 in which we pointed out that when the lifetime distribution does not depend on location the optimal survivor search plan ψ^* is obtained from the uniformly optimal search plan φ^* given in Theorem 2.2.4 or 2.4.6 of Stone (1976) as follows:

$$\psi^*(x, t) = \frac{\partial \varphi^*(x, t)}{\partial t} \quad \text{for } x \in X \text{ and } t \geq 0.$$

Example 2.2.1 of [5] finds B. O. Koopman's uniformly optimal search plan φ^* for the case of a circular normal target distribution and exponential detection function when $m_2(t) = Ut$ for $t \geq 0$ and U is a positive constant. The function φ^* has the form

$$\varphi^*(x, t) = \begin{cases} H\sqrt{t} - \dfrac{x_1^2 + x_2^2}{2\sigma^2} & \text{for } \dfrac{x_1^2 + x_2^2}{2\sigma^2} \leq H\sqrt{t}, \\ 0 & \text{otherwise,} \end{cases} \tag{3.17}$$

where $x = (x_1, x_2)$ and σ and H are positive constants. In this case, $\partial\varphi^*/\partial t$ is unbounded. Thus, the assumptions of Theorem 3.2 rule out this special case.

In practice, this problem can usually be avoided by leveling the prior target distribution in a small area about the point of highest probability density. However, for part (b), the sufficiency part of Theorem 3.2, a more satisfactory solution is available.

Suppose that the lifetime of the target is bounded; that is, there is a finite T such that

$$\alpha(x, T) = 1 \quad \text{for} \quad x \in X.$$

Let L be the set of real-valued Borel measurable functions f defined on $X \times [0, T]$ such that

$$\|f\| = \int_0^T \operatorname*{ess\,sup}_{x \in X} |f(x, t)| \, dt < \infty.$$

By identifying functions which are equal almost everywhere, L becomes a Banach space under the above norm. One can show under the assumptions of Theorem 3.2 that $F'(\psi, h)$ exists for all ψ and $h \in L$ and is given by (3.3). Thus, part (b) of Theorem 3.2 can be replaced by

(b′) *Suppose the lifetime of the target is bounded and $b(x, \cdot)$ is concave for $x \in X$. If there is a $\psi^* \in \Psi_2(m_2) \cap L$ such that*

$$\int_X \psi^*(x,s)\,dx = m_2(s) \quad \text{for a.e.} \quad s \in [0,\infty)$$

and a Borel function $\lambda : [0,\infty) \to [0,\infty)$ *such that* ψ^* *and* λ *satisfy* (3.2), *then* ψ^* *is an optimal survivor search plan within* $\Psi_2(m_2) \cap L$.

The proof of (b′) is identical to that of (b).

Note that if φ^* is given by (3.17), then for any $0 \le T < \infty$, $\partial\varphi^*/\partial t \in L$. In addition, $L_\infty \subset L$.

4. Example

In this section we illustrate the use of Theorem 3.2 by finding an optimal survivor search plan for a simple situation. Suppose the target distribution is specified by two regions, R_1 and R_2, of the plane both having area 1 and such that

$$p(x) = \begin{cases} p_1 & \text{for} \quad x \in R_1, \\ p_2 & \text{for} \quad x \in R_2, \\ 0 & \text{otherwise.} \end{cases}$$

The detection function is assumed to be homogeneous and exponential, i.e., there is a positive constant w such that

$$b(x,z) = 1 - e^{-wz} \quad \text{for} \quad x \in X \quad \text{and} \quad z \ge 0.$$

The target lifetime distributions are

$$\alpha(x,t) = 1 - e^{-t/\tau_1} \quad \text{for} \quad x \in R_1 \quad \text{and} \quad t \ge 0$$

$$\alpha(x,t) = 1 - e^{-t/\tau_2} \quad \text{for} \quad x \in R_2 \quad \text{and} \quad t \ge 0.$$

Observe that τ_1 and τ_2 are the mean target lifetimes given that the target is in regions 1 and 2, respectively. We assume $m_2(t) = 1$ and

$$\frac{p_1}{\tau_1} \ge \frac{p_2}{\tau_2}.$$

Consider two cases.

Case 1. $\tau_1^{-1} > \tau_2^{-1} + w$. For this case we show that the optimal survivor search plan is given as follows:

$$\psi^*(x,t) = \begin{cases} 1, & 0 \le t \le t_0 \\ 0, & t_0 < t < \infty, \end{cases} \quad \text{for} \quad x \in R_1,$$

$$\psi^*(x,t) = \begin{cases} 0, & 0 \le t \le t_0 \\ 1, & t_0 < t < \infty, \end{cases} \quad \text{for} \quad x \in R_2,$$

where

$$t_0 = [w + \tau_1^{-1} - \tau_2^{-1}]^{-1} \ln \left[\frac{p_1(1+w\tau_2)}{p_2}\right].$$

That is, the optimal plan is to perform all search in R_1 until the switch over time t_0. At this time search ceases in R_1 and switches over entirely to R_2 for the remainder of the search.

To show that ψ^* is optimal, we recall that

$$D(\psi^*, x, s) = p(x) \int_s^\infty \alpha'(x,t) b'\left(x, \int_0^t \psi^*(x,u)\mathrm{d}u\right)\mathrm{d}t \quad \text{for } x \in X \text{ and } s \ge 0.$$

Let

$$\varphi_j^*(t) = \int_0^t \psi^*(x,u)\mathrm{d}u \quad \text{for} \quad x \in R_j \quad \text{and} \quad t \ge 0.$$

Then

$$D(\psi^*, x, s) = \frac{wp_j}{\tau_j} \int_s^\infty \mathrm{e}^{-t/\tau_j}\, \mathrm{e}^{-w\varphi_j^*(t)}\mathrm{d}t \quad \text{for} \quad x \in R_j \quad \text{and} \quad s \ge 0.$$

Since $D(\psi^*, x, s)$ is constant over the region R_j for $j = 1, 2$, we may let

$$\lambda_j(s) = D(\psi^*, x, s) \quad \text{for} \quad x \in R_j \quad \text{and} \quad s \ge 0.$$

By taking

$$\lambda(s) = \begin{cases} \lambda_1(s) & \text{for} \quad 0 \le s \le t_0, \\ \lambda_2(s) & \text{for} \quad t_0 < s < \infty, \end{cases}$$

one may check that the conditions of Theorem 3.2 (b) are satisfied and that ψ^* is optimal.

Case 2. $\tau_1^{-1} \le \tau_2^{-1} + w$. Let

$$\theta = (\tau_2^{-1} - \tau_1^{-1})/w.$$

The optimal plan is given by

$$\psi^*(x,t) = \begin{cases} 1, & 0 \le t \le t_0 \\ \dfrac{1+\theta}{2}, & t_0 < t < \infty, \end{cases} \quad \text{for} \quad x \in R_1,$$

$$\psi^*(x,t) = \begin{cases} 0, & 0 \le t \le t_0 \\ \dfrac{1-\theta}{2}, & t_0 < t < \infty, \end{cases} \quad \text{for} \quad x \in R_2,$$

where

$$t_0 = \begin{cases} [w + \tau_1^{-1} - \tau_2^{-1}]^{-1} \ln\left(\dfrac{p_1\tau_2}{p_2\tau_1}\right) & \text{if} \quad w + \tau_1^{-1} > \tau_2^{-1}, \\ \infty & \text{otherwise.} \end{cases}$$

Observe that if $t_0 < \infty$ in case 2, then

$$\tau_2^{-1} - w < \tau_1^{-1} \le \tau_2^{-1} + w$$

so that $-w < \tau_1^{-1} - \tau_2^{-1} < w$ and $|\theta| \le 1$.

That ψ^* is optimal may be verified by checking that it satisfies conditions (3.2).

In order to understand why the optimal solution breaks into two cases, we consider the product

$$p(x)[1 - \alpha(x,t)] = p_j \, \mathrm{e}^{-t/\tau_j} \quad \text{for} \quad x \in R_j. \tag{4.1}$$

The expression in (4.1) is the probability that the target is in cell j and is alive by time t. This probability is decreasing exponentially at the rate τ_j^{-1}. If one were to place all his effort in cell j, then $p_j \, \mathrm{e}^{-t/\tau_j} \, \mathrm{e}^{-wt}$ would be proportional to the posterior probability that the target is in cell j and is alive by time t given it has not been found by time t. Thus, in case 1 when $\tau_1^{-1} > \tau_2^{-1} + w$, the posterior probability that the target is in region 1 is decreasing faster than the posterior probability that the target is in region 2 even if all search effort is placed in cell 2. The result is that the search starts in region 1 and eventually switches to region 2. When this switch does occur, region 1 will remain less attractive than region 2 throughout the remainder of the search. This results in an optimal plan which places all effort in region 1 until a switching time t_0 and then places all effort in region 2 from time t_0 onward.

In case 2, $\tau_1^{-1} \le \tau_2^{-1} + w$, and one is able to balance the rates of decrease by adjusting the rate at which search effort is placed in regions 1 and 2. In this case, the optimal plan places all effort in region 1 until time t_0 at which

$$p_1\alpha'(x,t_0)\,b'(x,t_0)=\frac{p_1}{\tau_1 w}\,\mathrm{e}^{-t_0(\tau^{-1}+w)}\quad\text{for}\quad x\in R_1$$

$$=\frac{p_2}{\tau_2 w}\,\mathrm{e}^{-t_0\tau_2^{-1}}$$

$$=p_2\alpha'(x,t_0)\,b'(x,0)\quad\text{for}\quad x\in R_2\,.$$

This is the t_0 given for case 2. From this time forward, effort is split between the two regions in such a way as to maintain $p_1\alpha'(x,t)b'(x,\varphi_1^*(t))=p_2\alpha'(y,t)b'(y,\varphi_2^*(t))$ for $t\geq t_0$, $x\in R_1$, and $y\in R_2$.

Note that in case 1, $\tau_2\tau_1^{-1}>1+w\tau_2$, so that t_0 for case 1 is always less than t_0 for case 2.

Acknowledgment

The survivor search problem was first formulated by H. R. Richardson in connection with the development of computer-assisted search planning (CASP) programs for the U. S. Coast Guard. He also suggested that the optimal survivor lifetime plan for the situation of the example would have the basic switch over form given there.

The use of the methods of Dubovitskii and Milyutin as given in [3] to obtain the functionals g_0, g_1, and g_2 of Theorem 3.1 follows Pursiheimo [4] up to the point of equation (3.9).

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