SIAM J. CONTROL
Vol. 12, No. 3, August 1974

NECESSITY AND EXISTENCE RESULTS ON CONSTRAINED OPTIMIZATION OF SEPARABLE FUNCTIONALS BY A MULTIPLIER RULE*

DANIEL H. WAGNER AND LAWRENCE D. STONE†

Abstract. Functionals E (extended real-valued) and C (vector-valued) are defined by $E(q) = \int_X e(x, q(x))\,d\mu x$ and $C(q) = \int_X c(x, q(x))\,d\mu x$, in a rather abstract setting, without differentiability assumptions. If, among other assumptions, either μ is a nonatomic measure or convexity conditions hold, then from Blackwell's generalization of Lyapunov's convexity theorem and a separating hyperplane theorem, it follows that for q^* to maximize E subject to an equality (inequality) constraint on C, it is necessary that there exist a vector λ (nonnegative vector λ) such that q^* maximizes $E - \lambda \cdot C$. For the latter to hold, by von Neumann's selection theorem under the principal condition that e and c are Borel functions, it is necessary that $q^*(x)$ maximize $e(x, \cdot) - \lambda \cdot c(x, \cdot)$ for a.e. $x \in X$. This much extends methods and results of Aumann and Perles. Existence results are derived from "upper closure" of the range of (C, E), under boundedness assumptions, as the range of a vector integral over selections from a set-valued function. This development utilizes and extends results and methods of Olech and Blackwell. The main result asserts existence of optimal functions under the principal conditions that the set-valued function is upper closed, e and c are Borel functions, C is bounded, and either μ is nonatomic or E is bounded above. Examples show that the results cannot be strengthened in various ways.

1. Introduction. This paper is addressed to functions q^* which maximize a real-valued "effectiveness" functional E subject to, e.g., a closed convex set constraint on a k-vector-valued "cost" functional C. Both E and C are separable (sum of point-functions): $E(q) = \int_X e(x, q(x))\,d\mu x$ and $C(q) = \int_X c(x, q(x))\,d\mu x$ when $q(x) \in Y(x)$ for $x \in X$, with fixed X, Y, μ, e, and c. We give general conditions under which such optimal q^* exist. We further show under weak conditions that it is necessary that for some vector λ, such optimal q^* must maximize the Lagrangian $E - \lambda \cdot C$ (a functional multiplier rule), and that for this it is necessary that $q^*(x)$ maximize the Lagrangian $e(x, \cdot) - \lambda \cdot c(x, \cdot)$ for a.e. $x \in X$ (a pointwise multiplier rule). In § 2, the formal framework is given and the relevance of the nonlinear forms of E and C is noted.

We make no differentiability assumptions, although in some cases convexity-concavity assumptions on $c(x, \cdot)$, $e(x, \cdot)$ occur as alternative hypotheses, implying one-sided differentiability. In contrast to related papers, we admit infinite values of E and C, Theorem 2.2 being a key tool for this purpose, and we impose relatively few continuity or boundedness conditions.

We use rather abstract measure-theoretic and topological assumptions, since this adds little difficulty. This has the advantages of identifying with greater precision the properties that are needed to insure the conclusions of the theorems and of enabling application to broader categories of situations, as compared, for example, to using Euclidean space with Lebesgue measure or counting measure (discrete summation). Such more familiar structures are related to some of the

* Received by the editors February 15, 1972, and in revised form April 24, 1973. This work was supported in part by the Naval Analysis Programs, Office of Naval Research, under Contract N00014-69-C-0435.

† Daniel H. Wagner, Associates, Paoli, Pennsylvania 19301.

relatively abstract hypotheses by comments in the introduction to § 5 and Remarks 2.3, 3.4 and 6.4.

Section 6 deals with existence, usually assuming bounded C and σ-finite μ. Theorem 6.3 (ii) and from Theorem 6.10 to the end of the paper constitute new results. The main result is Theorem 6.13 (iii); we use most of the paper in its proof. The principal new features of our existence results are in permitting ∞ to be attained, constraint sets to be rather general, and for $x \in X$, $Z(x) = \{(c(x, y), e(x, y)) : y \in Y(x)\} \subset \mathscr{E}_{k+1}$ to be "upper closed" instead of closed. In circumstances of interest (see Remark 6.4), upper closure permits $e(x, \cdot)$ to be upper semicontinuous instead of continuous. With $Z(x)$ closed for $x \in X$, no topology on X is needed (Theorem 6.9), but with the change to upper closure, Borel conditioning must be imposed (see Remark 6.11). Theorem 6.13 then asserts that the range of (C, E) is upper closed (which insures existence) if E is bounded above, and that existence holds if *either* μ is nonatomic or E is bounded above; Remark 6.14 shows that existence may fail if the latter disjunction is violated. Theorems 6.3 (i), 6.6, and 6.9 depend on $Z(x)$ being closed for $x \in X$ and are adaptations, with modest generalization, of work of Blackwell [5] and Olech [31], except that 6.9 depends on 6.3 (ii). A more systematic summary of § 6 is given in its introduction.

Statements of necessity of the functional multiplier rule for optimality are given in § 3, without topologizing X. Necessity of the pointwise rule is given in § 5. The main necessity result is Corollary 5.2; all of the prior theorems are used in its proof. In particular, the von Neumann selection result, Theorem 4.1, is needed, requiring Borel conditions similar to those of Theorem 6.13. That necessity of the pointwise rule may fail if e is not a Borel function but is Lebesgue measurable is shown in Remark 5.5. Theorem 5.3 gives necessity of the pointwise rule under countable X and convexity conditions.

The development leading to Corollary 5.2 is largely comprised of a Lyapunov-type convexity theorem, a separating hyperplane argument, and von Neumann's selection theorem; this pattern of proofs has also been previously used by Aumann and Perles [3] to show necessity of a pointwise multiplier rule for optimality, but under more restrictive conditions (see Remark 5.6). Our necessity results are thus largely adaptations of prior work, in more general settings. Remark 5.5 is new, the inclusion of infinite integrability appears to be new, and the distinction between pointwise and functional rules is at least not usual. The generality of our necessity statements is, moreover, needed in § 6 and in [37] and [38].

Theorems 3.1 and 6.6 (ii) extend Blackwell's generalizations [5] to vector integrals of Lyapunov's theorem [23] on the convexity and compactness of the range of a vector measure, a history of which is given in [39]. For related results see, for example, [1], [2], [4], [14]–[16], [27]–[32], and [34].

In a succeeding article [42], we prove necessity of the pointwise rule and existence under a "coverability" condition pertaining to the concave envelope of $e(x, \cdot)$, under $c(x, y) = y$, $y \in Y(x)$, $x \in X$. This avoids Borel assumptions and, in existence statements, replaces the upper closure condition with an alternative. It is interesting that coverability has a property similar to the property of Borel functions that composition with a measurable function results in a measurable function (see [42, Lemma 4 (iv)]), and that either coverability or Borel assumptions yield necessity of the pointwise rule.

Our results on functionals also apply to optimization of separable set-functions, by identifying sets with indicator functions. In this way, one can obtain the necessity and existence results of Dantzig and Wald [9] on the Neyman–Pearson lemma, removing the finite integrability restriction from their necessity statements.

2. Preliminaries. Let ω be the set of positive integers and $\varnothing$ be the empty set. For $n \in \omega$, let $\mathscr{E}_n$ be Euclidean n-space and $0^n = (0, \cdots, 0) \in \mathscr{E}_n$. If $a, b \in \mathscr{E}_n$, we denote their inner product by $a \cdot b$; this is extended to vectors with $\pm\infty$ components in the obvious way, being undefined if $0 \cdot \infty$ or $\infty - \infty$ occurs. If A is a subset of extended $\mathscr{E}_n$, we denote **fin** $A = A \cap \mathscr{E}_n$ (the finite part of A). A subscript on a vector or a vector-valued function will always denote component. For $B \subset \mathscr{E}_n$, we denote the closure, interior, and boundary of B by **cl** B, **int** B, and **bdy** B. The range of a function f is denoted **rng** f, u.s.c. means upper semicontinuous, and for a, b in extended $\mathscr{E}_n$, $a \leqq b$ means $a_i \leqq b_i$ for $i = 1, \cdots, n$. Let $\mathscr{E}_n^+ = \mathscr{E}_n \cap \{a : a \geqq 0^n\}$.

We fix arbitrary nonvacuous sets X and $Y(x)$ for each $x \in X$. Defining

$$\Omega = \{(x, y) : x \in X \text{ and } y \in Y(x)\},$$

we fix $k \in \omega$ and real-valued functions $c_1, \cdots, c_k$ and e on Ω. We denote $c = (c_1, \cdots, c_k)$ and $(c, e) = (c_1, \cdots, c_k, e)$, and we regard $\mathscr{E}_{k+1} = \mathscr{E}_k \times \mathscr{E}_1$.

We fix a measure μ over X. *Measurability always refers to* μ. *Integrability* of $f : X \to \mathscr{E}_n$ means that $\int_X f_i \, d\mu$ exists (possibly $\pm\infty$) for $i = 1, \cdots, n$. A measurable set S is an *atom* of μ if $\mu(S) > 0$ and S is not a union of disjoint measurable sets having positive measure. We say μ is *nonatomic* if it has no atoms, is *purely atomic* if X is a union of atoms, or has *finite substance* if for each measurable $P \subset X$ for which $\mu(P) > 0$, there exists a measurable $Q \subset P$ such that $0 < \mu(Q) < \infty$ (e.g., if μ is σ-finite). Subsets of X having μ measure zero are ignored, i.e., if $P \subset X$ and $Q \subset X$, then "for $x \in P$" means "for μ a.e. $x \in P$", and "$P = Q$" means "$\mu(P - Q) = \mu(Q - P) = 0$."

For $x \in X$, we denote by $c(x, \cdot)$ the function mapping y into $c(x, y)$ for $y \in Y(x)$; if $q(x) \in Y(x)$ for $x \in X$, $c(\cdot, q(\cdot))$ is the function mapping x into $c(x, q(x))$ for $x \in X$, and similarly for e. We define

$$\Psi = \{q : q \text{ is a function on } X \text{ and } q(x) \in Y(x) \text{ for } x \in X\},$$

$$\Xi = \Psi \cap \{q : c(\cdot, q(\cdot)) \text{ and } e(\cdot, q(\cdot)) \text{ are measurable functions}\},$$

$$\Phi = \Xi \cap \{q : c(\cdot, q(\cdot)) \text{ and } e(\cdot, q(\cdot)) \text{ are integrable}\},$$

$$C_i(q) = \int_X c_i(x, q(x)) \, d\mu x \quad \text{for } i = 1, \cdots, k, \quad q \in \Phi,$$

$$E(q) = \int_X e(x, q(x)) \, d\mu x \quad \text{for } q \in \Phi.$$

Denoting $C = (C_1, \cdots, C_k)$ and $(C, E) = (C_1, \cdots, C_k, E)$, we say $q^* \in \Phi$ is *optimal* if[1]

$$E(q^*) = \max \{E(q) : C(q) = C(q^*)\}$$

[1] In this and similar usage, it is understood that $E(p) \in \{E(q) : C(q) = C(q^*)\}$ implies that $E(p)$ exists.

and we say q^* is *strongly optimal* if

$$E(q^*) = \max \{E(q):C(q) \leqq C(q^*)\}.$$

Since q^* appears on both sides of the formulas defining optimality, these definitions do not refer to a preassigned cost; optimality (strong optimality) of q^* merely means that there is no $q \in \Phi$ such that $C(q) = C(q^*)(C(q) \leqq C(q^*))$ and $E(q) > E(q^*)$. In statements on existence of optimal functions, we shall, however, preassign cost by requiring $C(q) \in N$, with closed N.

An equality constraint may be considered to be an inequality constraint by noting that for $a, b \in \mathscr{E}_n$, $a = b$ iff $(a \leqq b$ and $-a \leqq -b)$.

Suppose $\lambda \in \mathscr{E}_k$. We define the *pointwise Lagrangian*, l_λ, by

$$l_\lambda(x, y) = e(x, y) - \lambda \cdot c(x, y) \quad \text{for } (x, y) \in \Omega,$$

and the *functional Lagrangian*, L_λ, by

$$L_\lambda(q) = E(q) - \sum_{\lambda_i \neq 0} \lambda_i C_i(q) \quad \text{for } q \in \Phi \text{ such that } \infty - \infty \text{ does not occur}.$$

Note that if $L_\lambda(q)$ exists, then $L_\lambda(q) = \int_X l_\lambda(x, q(x))\, d\mu x$. Suppose $q^* \in \Phi$. We say that (q^*, λ) *satisfies the functional multiplier rule* if $L_\lambda(q^*)$ exists and

$$L_\lambda(q^*) \geqq L_\lambda(q) \quad \text{whenever } L_\lambda(q) \text{ exists}. \tag{2.1}$$

We say that (q^*, λ) *satisfies the pointwise multiplier rule* if

$$l_\lambda(x, q^*(x)) \geqq l_\lambda(x, y) \quad \text{for } y \in Y(x), \quad x \in X. \tag{2.2}$$

If either rule is satisfied with $\lambda \in \mathscr{E}_k^+$, we say the rule is *strongly* satisfied.

Traditionally, Lagrange multiplier rules are stated in terms of finding a stationary point of a Lagrangian, requiring differentiability assumptions in contrast to the formulations above. The usefulness of the present viewpoint, maximizing a Lagrangian without differentiability assumptions, was pointed up by Everett [10]. Earlier statements of multiplier rules in this vein were given by Slater [36] and by Lehmann as Lemma 3 in [21, § 3.6]. For more history, see [22], [12] and [41].

We now define an additional set-valued function Z by

$$Z(x) = \{(c(x, y), e(x, y)) : y \in Y(x)\} \quad \text{for } x \in X.$$

Note that for $x \in X$, while $Z(x) \subset \mathscr{E}_{k+1}$, $Y(x)$ is an arbitrary set. We fix $\mathscr{F}$ as the set of integrable functions f on X such that $f(x) \in Z(x)$ for $x \in X$ and define

$$I_i(f) = \int_X f_i(x)\, d\mu x \quad \text{for } f \in \mathscr{F}, \qquad i = 1, \cdots, k+1,$$

and

$$I = (I_1, \cdots, I_{k+1}).$$

By the Axiom of Choice, **rng** $I =$ **rng** (C, E). The nonlinear (C, E) and arbitrarily-valued Y provide a more general framework than that of the linear I. We shall subsequently condition e and c in ways which yield results not obtainable from I alone.

THEOREM 2.1. *If* $\lambda \in \mathscr{E}_k$, $q^* \in \Phi$, $E(q^*) > -\infty$, *and* $C(q^*)$ *is finite, then in the following,* (i) *is sufficient for* (ii) *and* (ii) *is sufficient for* (iii):

(i) (q^*, λ) *satisfies (strongly satisfies) the pointwise multiplier rule;*

(ii) (q^*, λ) *satisfies (strongly satisfies) the functional multiplier rule;*

(iii) q^* *is optimal (strongly optimal).*

Proof. This is straightforward, with care to avoid $\infty - \infty$.

THEOREM 2.2. *Suppose* μ *has finite substance,* $q^* \in \Phi$, $E(q^*)$ *and* $C(q^*)$ *are finite, and* $\lambda \in \mathscr{E}_k$. *Then the following three conditions are equivalent:*

(i) $L_\lambda(q^*) \geqq L_\lambda(q)$ *whenever* $L_\lambda(q)$ *exists;*

(ii) $l_\lambda(x, q^*(x)) \geqq l_\lambda(x, q(x))$ *for* $x \in X$, *whenever* $q \in \Xi$;

(iii) $L_\lambda(q^*) \geqq L_\lambda(q)$ *whenever* $E(q)$ *and* $C(q)$ *are finite.*

Proof. Obviously (i) implies (iii), and (ii) implies (i). If (iii) holds and (ii) fails, then for some $q \in \Xi$, letting $P = \{x : l_\lambda(x, q(x)) > l_\lambda(x, q^*(x))\}$, we have $\mu(P) > 0$. For $j \in \omega$, let

$$P_j = P \cap \{x : |e(x, q(x))| \leqq j \text{ and } |c_i(x, q(x))| \leqq j,\ i = 1, \cdots, k\}.$$

Then $\bigcup_{j=1}^{\infty} P_j = P$, so for some j_0, $\mu(P_{j_0}) > 0$. Take a measurable $S \subset P_{j_0}$ such that $0 < \mu(S) < \infty$. Then $e(\cdot, q(\cdot))$ and $c(\cdot, q(\cdot))$ restricted to S are finitely integrable. Define $p(x) = q(x)$ for $x \in S$ and $p(x) = q^*(x)$ for $x \in X - S$. Then $p \in \Phi$ and $L_\lambda(p) > L_\lambda(q^*)$ in contradiction, which completes the proof.

Remark 2.3. In Remark 3.4, Theorem 5.3, and Theorem 6.3, X is countable and $\mu(\{x\}) = 1$ for $x \in X$, whence $E(q) = \sum_{x \in X} e(x, q(x))$, and similarly for C. These discrete summation byproducts of integration results also hold if summation is defined as a limit of partial sums.

3. Necessity of functional multiplier rule for optimality. Corollary 3.3 below gives the necessity of the (strong) functional multiplier rule for (strong) optimality. Theorem 3.2 asserts necessity of a Kuhn–Tucker [19] condition, (iii), for strong optimality for a given cost. These follow from a known consequence of a separating hyperplane theorem, applied with Theorem 2.2 and a generalized Lyapunov convexity statement, Theorem 3.1. Theorem 3.2 ((ii) iff (iv)) has been given by Meeks and Francis [26], [25] under stronger concavity and other conditions.

We say that the functionals C *and* $-E$ *are convex* if

$$\hat{\Phi} = \Phi \cap \{q : C_i(q) < \infty \text{ for } i = 1, \cdots, k \text{ and } E(q) > -\infty\}$$

is a convex subset of a real vector space and whenever $0 \leqq \alpha \leqq 1$ and $q, p \in \hat{\Phi}$, we have $C(\alpha q + (1 - \alpha)p) \leqq \alpha C(q) + (1 - \alpha)C(p)$ and $E(\alpha q + (1 - \alpha)p) \geqq \alpha E(q) + (1 - \alpha)E(p)$.

THEOREM 3.1 (Lyapunov and Blackwell). *If* μ *is nonatomic, then* **fin rng** I, *i.e.,* **fin rng** (C, E), *is convex.*

Proof. Apply the proof of Theorem 3 of [5].

THEOREM 3.2. *Suppose* μ *has finite substance,* $v \in \mathscr{E}_k$, $u \in$ **rng** C, $u_i < v_i$ *for* $i = 1, \cdots, k$, $q^* \in \Phi$, $E(q^*)$ *and* $C(q^*)$ *are finite, and* $C(q^*) \leqq v$. *Suppose also that either* (a) μ *is nonatomic or* (b) C *and* $-E$ *are convex. Then the following are equivalent:*

(i) $E(q^*) = \max\{E(q) : C(q) \leqq v\}$;

(ii) $E(q^*) = \max\{E(q) : -\infty < C_i(q) \leqq v_i, i = 1, \cdots, k\}$;

(iii) *there exists* $\lambda \in \mathscr{E}_k^+$ *such that whenever* $q \in \Phi$ *and* $L_\lambda(q)$ *exists,*

(3.1a) $$L_\lambda(q) + \lambda \cdot v \leqq L_\lambda(q^*) + \lambda \cdot v \leqq L_\eta(q^*) + \eta \cdot v \quad \text{for } \eta \in \mathscr{E}_k^+,$$

and

$$\lambda \cdot [v - C(q^*)] = 0; \tag{3.1b}$$

(iv) *there exists a* $\lambda \in \mathscr{E}_k^+$ *such that* (3.1) *holds when* $E(q)$ *and* $C(q)$ *are finite.*

Proof. Clearly (i) implies (ii); by Theorem 2.2 ((i) iff (iii)), we have (iii) iff (iv); and the proof that (iii) implies (i) is straightforward. That (ii) implies (iv) is a corollary (as pointed out by a referee) of Luenberger's Theorem 1 [22, p. 217] and its corollary by choosing the Ω of [22] to be **fin** $\{(w, d): w \geqq C(q) \text{ and } d \leqq E(q)$ for some $q \in \Phi\}$; this is easily shown to be convex, using Theorem 3.1 in case (a).

COROLLARY 3.3. *Suppose* μ *has finite substance,* $q^* \in \Phi$, $|E(q^*)| < \infty$, $C(q^*) \in$ **int rng** C, *and either* (a) μ *is nonatomic or* (b) C *and* $-E$ *are convex. Then for* q^* *to be optimal (strongly optimal) it is necessary and sufficient that for some* $\lambda \in \mathscr{E}_k$, (q^*, λ) *satisfy (strongly satisfy) the functional multiplier rule.*

Proof. This follows from Theorem 3.2 by setting $v = C(q^*)$ and noting that q^* is optimal iff $E(q^*) = \max\{E(q): C(q) \leqq v \text{ and } -C(q) \leqq -v\}$.

Remark 3.4. Hypothesis (b) in Theorem 3.2 and Corollary 3.3 is satisfied if for $x \in X$, $Y(x)$ is a convex subset of a real vector space and $c(x, \cdot)$ and $-e(x, \cdot)$ are convex functions. This is useful when X is countable and $\mu(\{x\}) = 1$ for $x \in X$ (so integration is discrete summation; see Remark 2.3), since hypothesis (a) does not then apply. That necessity in Theorem 3.2 and Corollary 3.3 fails in this discrete case, in the absence of convexity of $-e(x, \cdot)$, $x \in X$, is shown by example in [37] and [43].

Remark 3.5. Suppose Φ_0 is a subset of Φ which is closed under "switching" or "exchange" in the sense that if $q, p \in \Phi_0$, P is measurable, $r(x) = p(x)$ for $x \in P$, and $r(x) = q(x)$ for $x \in X - P$, then $r \in \Phi_0$. Then Theorems 2.2, 3.1, and 3.2 and Corollary 3.3 hold if in the definitions and statements involving Φ, Φ_0 is used instead.

4. Von Neumann selection theorem. We now make topological assumptions, under which we give a generalized version of von Neumann's selection result, needed in § 5 and § 6. This is given as Lemma 5 of [40] under more special conditions, e.g., $X = \mathscr{E}_1$; however, von Neumann's method of proof suffices also for the present version. A new proof is given here, due to Professor Herbert Federer and based on the proof of § 2.2.12 of [11]. Professor J. C. Oxtoby has provided an alternative formulation of this proof, also outlined below.

If A and B are topological spaces and $f: A \to B$, we say that f is a *Borel function* if for each open $G \subset B$, $f^{-1}(G)$ is a Borel subset of A. If also g is a measurable function into A, then $f \circ g$ is measurable. Let $\mathscr{N} = \omega^\omega$, the set of infinite sequences of positive integers with the product topology formed from the discrete topology on each factor ω; $\mathscr{N}$ is homeomorphic to the irrational numbers. Any continuous image of $\mathscr{N}$ in a Hausdorff space is a Suslin set. Every Borel subset of a metric space is a Suslin set. If open sets are measurable, so are Suslin sets. The foregoing is found, for example, in § 2.2 of [11].

THEOREM 4.1 (von Neumann). *Suppose* X *is a Hausdorff space, open subsets of* X *are* μ *measurable,* $\mathscr{S}$ *is a Suslin subset of a complete separable metric space* W, $h: \mathscr{S} \to X$ *is continuous,* $\mu(h(\mathscr{S})) < \infty$, *and* $\varepsilon > 0$. *Then there exist a compact* $D \subset h(\mathscr{S})$ *and a Borel function* $f: D \to \mathscr{S}$ *such that* $\mu(h(\mathscr{S}) - D) \leqq \varepsilon$, *and* $h(f(x)) = x$ *for* $x \in D$.

Proof (Federer). For $S \subset \mathscr{S}$, define $\gamma(S) = \mu(h(S))$. (This uses the measure foundations of [11]; if [17] is used, one replaces μ by its associated outer measure.) Let g map $\mathscr{N}$ continuously onto $\mathscr{S}$ (see § 2.2.10 of [11]).

Define $Z_0 = (W \times \mathscr{N}) \cap \{(w, \sigma): w = g(\sigma)\}$. Then Z_0 is closed. For $(w, \sigma) \in W \times \mathscr{N}$ define $\eta(w, \sigma) = w$. Then $\eta(Z_0) = \mathscr{S}$. Corresponding to $i \in \omega$, we inductively choose $\tau_i \in \omega$ and closed sets $Z_i \subset Z_0$ so that

$$(4.1) \quad Z_i = Z_{i-1} \cap \{(w, \sigma): \sigma_i \leqq \tau_i\} \quad \text{and} \quad \gamma[\eta(Z_{i-1})] - \gamma[\eta(Z_i)] < \varepsilon 2^{-i};$$

this is possible because defining $A_{ij} = \eta(Z_{i-1} \cap \{(w, \sigma): \sigma_i \leqq j\})$ for $i, j \in \omega$, we have

$$\bigcup_{j=1}^{\infty} A_{ij} = \eta(Z_{i-1}) \quad \text{and} \quad \gamma\left(\bigcup_{j=1}^{\infty} A_{ij}\right) = \mu\left(\bigcup_{j=1}^{\infty} h(A_{ij})\right) = \lim_{j \to \infty} \gamma(A_{ij}).$$

Here we have used the fact that a continuous image in W or X of a Suslin subset of a complete separable metric space is also a Suslin set (see § 2.2.10 of [11]); hence each A_{ij} and $h(A_{ij})$ is a Suslin set, and each $h(A_{ij})$ is μ measurable.

Let $K = \mathscr{N} \cap \{\sigma: \sigma_i \leqq \tau_i \text{ for } i \in \omega\}$. Then K is compact (by Tikhonov's theorem) and

$$\bigcap_{i=1}^{\infty} Z_i = Z_0 \cap (W \times K) = (W \times \mathscr{N}) \cap \{(w, \sigma): \sigma \in K \text{ and } w = g(\sigma)\}.$$

Hence $g(K) = \eta(Z_0 \cap (W \times K)) \subset \mathscr{S}$. Let $D = h(g(K))$. Then $g(K)$ and D are compact.

There exists a Cantor set $\Gamma \subset [0, 1]$ and a continuous map β on Γ onto $g(K)$. Following [35, § 7.1, Chap IX], for $x \in D$, let $\alpha(x) = \min\{t: h(\beta(t)) = x\}$, whence $\alpha(x) \in \Gamma$. Let $f = \beta \circ \alpha$. For $t \in \Gamma$, $\{x: \alpha(x) \leqq t\} = h(\beta(\Gamma \cap [0, t]))$ which is a Borel set since $h \circ \beta$ is continuous. Thus, α is a Borel function and, therefore, so is f.

It remains to show that $\mu(h(\mathscr{S}) - h(g(K))) \leqq \varepsilon$. This will follow from (4.1) if we show that $g(K) \supset \bigcap_{i=1}^{\infty} \eta(Z_i)$ (the reverse inclusion is obvious), which may be proved by following exactly the argument in § 2.2.12 of [11] that $C \subset p[Z_0 \cap (X \times K)]$ (here W, η, and $\bigcap_{i=1}^{\infty} \eta(Z_i)$ correspond respectively to X, p, and C in [11]). This completes the proof.

An outline of Oxtoby's alternative formulation is as follows: take γ, g, and K as before, and choose $\tau \in \mathscr{N}$ and $\mathscr{N} = V_0 \supset V_1 \supset \cdots$ such that for $i \in \omega$,

$$V_i = V_{i-1} \cap \{\sigma: \sigma_i \leqq \tau_i\} \quad \text{and} \quad \gamma[g(V_{i-1})] - \gamma[g(V_i)] < \varepsilon 2^{-i};$$

then $K = \bigcap_{i=1}^{\infty} V_i$. Show that $g(K) \supset \bigcap_{i=1}^{\infty} g(V_i)$ by a diagonal selection argument similar to Sierpinski's, given in [35, § 5.3, Chap. II]. Then $D = h(g(K))$ serves as before.

Remark 4.2. If in Theorem 4.1 we require μ to be Borel regular (§ 2.2.3 of [11]) and X to be a metric space, then by Lusin's theorem (§ 2.3.5 of [11]) we can obtain f to be continuous. If μ is Borel regular and σ-finite, then any measurable function is a.e. equal to a Borel function (§ 2.3.6 of [11]); in that case, we can obtain f to be a Borel function defined a.e. on $h(\mathscr{S})$, without requiring $\mu(h(\mathscr{S})) < \infty$. For related results, see [6]–[8], [18], [20], [24], and [33].

5. Necessity of pointwise multiplier rule for optimality. We now apply the above results to prove the necessity of the pointwise multiplier rule for optimality, under some weak assumptions primarily of a Borel nature.

Defining $\pi(x, y) = x$ for $(x, y) \in \Omega$, we stipulate the following condition (also used in Lemma 6.12 and Theorem 6.13 below):

Condition (α): *X is a metric space, Ω is a Borel subset of a complete separable metric space, π is continuous, c and e are Borel functions, and μ is Borel regular.*

Note that Condition (α) is satisfied if, in particular, Ω is a Borel subset of $\mathscr{E}_n$, π projects Ω into $\mathscr{E}_m$ with $m < n$, c and e are Borel functions, and μ is m-dimensional Lebesgue measure. It also holds if $X = \omega$ and for $x \in X$, $\mu(\{x\}) = 1$ and $Y(x)$ is a Borel subset of $\mathscr{E}_k$.

THEOREM 5.1. *Suppose Condition* (α) *holds, μ has finite substance, $q^* \in \Phi$, $E(q^*)$ and $C(q^*)$ are finite, and $e(\cdot, q^*(\cdot))$ and $c(\cdot, q^*(\cdot))$ are Borel functions. Then for $\lambda \in \mathscr{E}_k$, for (q^*, λ) to satisfy (strongly satisfy) the functional multiplier rule, it is necessary and sufficient that (q^*, λ) satisfy (strongly satisfy) the pointwise multiplier rule.*

Proof. Sufficiency follows from Theorem 2.1.

Suppose (2.1) holds but (2.2) fails. Let

$$\mathscr{B} = \{(x, y): l_\lambda(x, q^*(x)) < l_\lambda(x, y)\}.$$

Then $\mathscr{B}$ is a Borel set, so $\pi(\mathscr{B})$ is measurable. Since (2.2) fails, $\mu(\pi(\mathscr{B})) > 0$. Choose a measurable $A \subset \pi(\mathscr{B})$ such that $0 < \mu(A) < \infty$. Since μ is Borel regular, there exists a Borel set $Q \supset A$ such that $\mu(Q) = \mu(A)$. Let $\mathscr{S} = \mathscr{B} \cap \pi^{-1}(Q)$. Then $\mathscr{S}$ is a Borel set and $0 < \mu(\pi(\mathscr{S})) < \infty$.

By Theorem 4.1 (with $h = \pi$), there exists a Borel set $P \subset \pi(\mathscr{B})$ and a Borel function $f: P \to \mathscr{B}$ such that $\pi(f(x)) = x$ for $x \in P$ and $\mu(P) > 0$. Let $\kappa(x, y) = y$ for $(x, y) \in \Omega$ and let $p = \kappa \circ f$. Then for $x \in P$, $(x, p(x)) \in \mathscr{B}$, i.e., $l_\lambda(x, p(x)) > l_\lambda(x, q^*(x))$. Let $\hat{q}(x) = p(x)$ for $x \in P$ and $\hat{q}(x) = q^*(x)$ for $x \in X - P$. Since p, e, and c are Borel functions, $\hat{q} \in \Xi$ and Theorem 2.2 (ii) fails with $q = \hat{q}$. Hence, Theorem 2.2 (i) fails, contrary to hypothesis.

COROLLARY 5.2. *Suppose the hypothesis of Theorem* 5.1 *holds, $C(q^*) \in$* **int rng** *C, and either* (a) *μ is nonatomic or* (b) *C and $-E$ are convex* (see § 3). *Then for q^* to be optimal (strongly optimal), it is necessary and sufficient that for some $\lambda \in \mathscr{E}_k$, (q^*, λ) satisfy (strongly satisfy) the pointwise multiplier rule.*

Proof. This follows from Corollary 3.3 and Theorem 5.1.

THEOREM 5.3. *Suppose X is countable and for $x \in X$, $\mu(\{x\}) = 1$, $Y(x) \subset \mathscr{E}_m$, $Y(x)$ is convex, and $c(x, \cdot)$ and $-e(x, \cdot)$ are convex functions. Suppose $q^* \in \Phi$, $|E(q^*)| < \infty$, and $C(q^*) \in$* **int rng** *C. Then the conclusion of Corollary* 5.2 *holds.*

Proof. Apply Corollary 3.3, Remark 3.4, and the method of proving Theorem 5.1 without resorting to Theorem 4.1.

Remark 5.4. If μ is σ-finite, then by Remark 4.2, Theorem 5.1 and Corollary 5.2 hold without assuming $e(\cdot, q^*(\cdot))$ and $c(\cdot, q^*(\cdot))$ are Borel functions.

Theorem 5.1 and Corollary 5.2 have *linear* functional corollaries which are easily formed by letting $Y = Z$, and for $(x, y) \in \Omega$, $(c(x, y), e(x, y)) = y$. The Borel condition on Ω becomes simply the condition that the "graph" of Z, i.e., $\{(x, z): x \in X, z \in Z(x)\}$, be a Borel subset of $\bar{X} \times \mathscr{E}_{k+1}$, where $\bar{X}$ is a completion of X, a separable metric space. The latter condition is not implied by the hypothesis of Theorem 5. Thus, Theorem 5.1 is not a corollary of the linear functional statement.

Remark 5.5. We show by example that the assumption in Theorem 5.1 (and Corollary 5.2) that e is a Borel function may not be replaced by the assumption that e is a measurable function with respect to a well-behaved measure over Ω.

Let $X = [0, 1]$ and let μ be the Lebesgue measure restricted to X. For $x \in X$, let $Y(x) = [-2, 2]$; thus, $\Omega = [0, 1] \times [-2, 2]$. Let μ^* and μ_* be, respectively, outer and inner Lebesgue measure on X (under the foundations of [11], $\mu = \mu^*$). Choose $D \subset X$ such that $\mu^*(D) = 1$ and $\mu_*(D) = 0$; then $\mu^*(X - D) = 1$, $\mu_*(X - D) = 0$, and D is not measurable. Let $k = 1$, let $c(x, y) = y$ for $(x, y) \in \Omega$, and defining

$$\mathscr{A} = \left\{(x, y): \left[x \in D \text{ and } |y| = 1 - \frac{1}{2i}\right] \text{ or } \left[x \in X - D \text{ and } |y| = 1 - \frac{1}{1 + 2i}\right] \text{ for some } i \in \omega\right\},$$

let e be the indicator function of $\mathscr{A}$. Then e is a measurable function with respect to two-dimensional Lebesgue measure, since $(\mu^* \times \mu^*)(\mathscr{A}) = 0$. However, $\pi(\mathscr{A} \cap ([0, 1] \times \{\tfrac{1}{2}\})) = D$, so $\mathscr{A}$ is not a Borel set, and e is not a Borel function.

Suppose $q \in \Phi$ with $E(q) > 0$. Then $e(\cdot, q(\cdot))$ and q, i.e., $c(\cdot, q(\cdot))$, are measurable functions. Let $P = \{x : e(x, q(x)) > 0\}$, $A = \{q(x) : x \in P\}$, and $S_y = \{x : q(x) = y\}$ for $y \in A$. For $y \in A$, $S_y \subset D$ or $S_y \subset X - D$, and, therefore, $\mu_*(S_y) = 0$, whence $\mu(S_y) = 0$, since S_y must be measurable. Therefore, $\mu(P) = 0$, since A is countable, in contradiction to $E(q) > 0$. Thus for no $q \in \Phi$ is $E(q) > 0$.

Let $q^*(x) = 0$ for $x \in X$. Then $C(q^*) = 0 = E(q^*)$ and $E(q^*) = \max\{E(q): C(q) = 0\}$. Thus q^* is optimal and $(q^*, 0)$ satisfies the functional multiplier rule. Clearly, (q^*, λ) does not satisfy the pointwise multiplier rule for any λ. Furthermore, there is no $(\hat{q}, \lambda)$ satisfying the pointwise multiplier rule with $C(\hat{q}) = C(q^*)$. Thus, Theorem 5.1 fails if e is not a Borel function, since all other hypotheses are satisfied.

If we redefine $e(x, -1) = e(x, 1) = 1$ for $x \in X$, then e is "coverable" as defined in [42] and the necessity conclusions of Theorem 5.1 and Corollary 5.2 are restored.

Remark 5.6. More restricted versions of Corollary 5.2 (with hypothesis (a)) have been obtained by Zahl [43], Aumann and Perles [3], and Meeks and Francis [26], [25], each using Lebesgue measure for μ. Although none of these is a corollary of the other two, among them the treatment with the most strength is in [3]. Indeed the pattern of our development leading to Corollary 5.2 is similar to the pattern used in [3], as noted in § 1. However, the version in [3] unnecessarily assumes that for $x \in X$, $Y(x) = \mathscr{E}_1^+$, and $e(x, \cdot)$ is nondecreasing and nonnegative.

Theorem 5.1 will be applied in proving our main result, Theorem 6.13 (iii). In [37] and [38], Stone applies Corollaries 5.2 and 5.3 to prove that, under weak hypotheses, incrementally optimal separable allocations are totally optimal.

6. Results on existence and rng I being upper closed. We now give results on existence of optimal functions, for cost constrained to a given closed set. Results of Olech [31], [32] and Blackwell [5] are of fundamental importance to this development; see Remark 6.7. Lemma 6.1, stated without proof, obviates explicit mention of the desired existence in most subsequent theorems by asserting that it follows from the finite part of the range of a separable vector functional, **fin rng** I (= **fin rng** (C, E)), being suitably bounded and "upper closed" (defined below).

We usually assume below that **rng** C is bounded and μ is σ-finite.

Upper closure results when μ is purely atomic, i.e., in effect when X is countable, are given in Theorem 6.3. Theorem 6.6 (ii) generalizes Lyapunov's compactness theorem and gives the desired existence when **rng** (C, E) is bounded and $Z(x)$ is closed for $x \in X$. Boundedness of **rng** E is weakened to **rng** E being bounded above in Theorem 6.9. Theorem 6.10 asserts that if for one p, $E(p) = \infty$, then, mainly under nonatomic μ, every interior C value is attained by a q with $E(q) = \infty$. For this much, no topology on X is needed.

Suppose for $x \in X$, $Z(x)$ is merely "upper closed" instead of closed. Then Theorems 6.6 and 6.9 fail, as shown by example in Remark 6.11. However, Theorem 6.13, which includes Condition (α) of § 5 (Borel and other topological assumptions), asserts existence if the constraint set is convex and either (a) μ is nonatomic or (b) **rng** E is bounded above. Remark 6.14 shows by example that existence may fail if both (a) and (b) fail. Theorem 6.13 also asserts that **fin rng** (C, E) is upper closed if (b) holds and that **fin rng** (C, E) contains the extreme points of its "upper boundary" if both (a) and (b) hold.

Suppose $A \subset \mathscr{E}_{k+1}$. We say A is *upper closed* if for $(w, d) \in \mathbf{cl}\, A$, there exists $b \geqq d$ such that $(w, b) \in A$. This specializes the concept given by Olech in [31] as lower closure with respect to a given closed convex proper cone, here taking the cone to be $\{0^k\} \times \{a : a \leqq 0\}$. We define the *upper boundary* of A, denoted **upbdy** A, by

$$\mathbf{upbdy}\, A = \mathbf{cl}\, A \cap \{(w, d) : d \geqq d' \text{ whenever } (w, d') \in \mathbf{cl}\, A\}.$$

If $\{d : (w, d) \in A \text{ for some } w\}$ is bounded above, then **upbdy** A is the graph of an u.s.c. function on $\{w : (w, d) \in A \text{ for some } d\}$.

For $w \in \mathbf{rng}\, C$, we define

$$v(w) = \sup \{E(q) : C(q) = w\}.$$

This supremum is attained iff there exists an optimal function with cost w.

We say b is an *extreme point* of $S \subset \mathscr{E}_{k+1}$ if $b \in S$ and there exist no $a, d \in S - \{b\}$ and $0 < \alpha < 1$ such that $b = (1 - \alpha)a + \alpha d$. By **ext** S we mean the set of extreme points of S.

LEMMA 6.1. *If* **rng** C *is bounded,* **rng** E *is bounded above,* **fin rng** (C, E) *is upper closed,* $N \subset \mathscr{E}_k$ *is closed, and* $N \cap \mathbf{rng}\, C \neq \emptyset$, *then* v *is u.s.c., and there exists* $p^* \in \Phi$ *such that* $C(p^*) \in N$ *and* $E(p^*) = \max \{E(p) : C(p) \in N\}$.

LEMMA 6.2. *Let* $g : X \to \mathscr{E}_{k+1}$ *be a measurable function and* $g(x) \in Z(x)$ *for* $x \in X$. *Suppose* $1 \leqq j \leqq k + 1$, $I_j(h) < \infty$ *for* $h \in \mathscr{F}$, *and* **fin rng** $I \neq \phi$. *Then* $\int_X g_j \, d\mu < \infty$.

Proof. We may take $j = k + 1$. Suppose the conclusion fails, and choose $f \in \mathscr{F}$ such that $I(f)$ is finite. Let $h^0 = g$. For $i = 1, \cdots, k$, inductively let $P_i = \{x : h_i^{i-1}(x) \geqq 0\}$, choose a measurable Q_i such that $Q_i \subset P_i$ or $Q_i \subset X - P_i$ and $\int_{Q_i} h_{k+1}^{i-1} \, d\mu = \infty$, and define $h^i(x) = h^{i-1}(x)$ for $x \in Q_i$ and $h^i(x) = f(x)$ for $x \in X - Q_i$; then h_n^i is integrable, for $n = 1, \cdots, i$, and $\int_X h_{k+1}^i \, d\mu = \infty$. Thus $h^k \in \mathscr{F}$ and $I_{k+1}(h^k) = \infty$, which contradicts the hypothesis.

THEOREM 6.3. *Suppose μ is σ-finite and purely atomic. Then:*
(i) *if $Z(x)$ is closed for $x \in X$ and* **rng** I (=**rng** (C, E)) *is bounded, then* **rng** I *is compact* (due largely to Blackwell [5]);
(ii) *if $Z(x)$ is upper closed for $x \in X$,* **rng** C *is bounded, and* **rng** E *is bounded above, then* **fin rng** (C, E) *is upper closed.*

Proof. We may assume **fin rng** $I \neq \varnothing$, X is countable, and $\mu(\{x\}) = 1, x \in X$. Suppose the hypothesis of (ii) holds and $(w, d) \in$ **cl fin rng** (C, E). We must find $g \in \mathscr{F}$ such that $(I_1(g), \cdots, I_k(g)) = w$ and $I_{k+1}(g) \geqq d$. Choose $h^1, h^2, \cdots \in \mathscr{F}$ such that $I(h^n) \to (w, d)$. Now $(h^1, h^2, \cdots)$ is a pointwise bounded sequence since $(w, d) \in \mathscr{E}_{k+1}$, **rng** $(I_1, \cdots, I_k)$ is bounded, and **rng** I_{k+1} is bounded above; by diagonal selection, choose a subsequence $(h^{\beta_1}, h^{\beta_2}, \cdots)$ which converges pointwise, to h by definition. For $x \in X, h(x) \in$ **cl** $Z(x)$, so since $Z(x)$ is upper closed, we may choose $b(x) \geqq h_{k+1}(x)$ such that $g(x) \equiv (h_1(x), \cdots, h_k(x), b(x)) \in Z(x)$.

For $x \in X$, by the upper closure of $Z(x)$ and boundedness conditions, we choose $m(x) \in Z(x)$ such that $m_{k+1}(x) \geqq z_{k+1}$ for $z \in Z(x)$. By Lemma 6.2, $|\int_X m_{k+1}\, d\mu| < \infty$. By Fatou's lemma applied to $(m_{k+1} - h^{\beta_1}_{k+1}, m_{k+1} - h^{\beta_2}_{k+1}, \cdots)$,

$$d = \lim_{n\to\infty} \int_X h^{\beta_n}_{k+1}\, d\mu \leqq \int_X \limsup_{n\to\infty} h^{\beta_n}_{k+1}(x)\, d\mu x = \int_X h_{k+1}\, d\mu \leqq \int_X b\, d\mu.$$

By similar argument, $w_i \leqq \int_X h_i\, d\mu \leqq w_i$ for $i = 1, \cdots, k$, proving (ii).

By similar but easier argument, one proves (i) to complete the proof.

Remark 6.4. The hypothesis "$Z(x)$ is upper closed for $x \in X$" appearing in Theorem 6.3 (ii) (and also in Theorem 6.13 below) is obviously satisfied if for $x \in X$, $Y(x)$ is a compact space, $c(x, \cdot)$ is continuous, and $e(x, \cdot)$ is u.s.c.

LEMMA 6.5 (Olech [31], [32]). *Suppose μ is nonatomic and σ-finite, and $b \in$* **ext cl fin rng** I. *Then there exist $f^1, f^2, \cdots \in \mathscr{F}$ and $f: X \to \mathscr{E}_{k+1}$ such that $f^j(x) \to f(x)$ for $x \in X$ and $\int_X f\, d\mu = b$.*

THEOREM 6.6. *Suppose μ is σ-finite. Then* (recall **rng** I = **rng** (C, E)):
(i) *if μ is nonatomic and $Z(x)$ is closed for $x \in X$, then* **fin rng** I *contains* **ext cl fin rng** I (due to Olech [31], [32]);
(ii) *if $Z(x)$ is closed for $x \in X$ and* **rng** I *is bounded, then* **rng** I *is compact.*

Proof. From Lemma 6.5, (i) follows. One proves (ii) by partitioning X into purely atomic and nonatomic subsets as in [5], and applying (i) and Theorem 6.3 (i).

Remark 6.7. Theorems 6.3 (i) and 6.6 (ii) were given by Blackwell [5] for the case where $\mu(X) < \infty$ and $Z(x)$ is the same for each $x \in X$. Lemma 6.5 and Theorem 6.6 (i) are due to Olech as noted (see Lemma 2, Theorem 3, and related discussion in [31]). Also in [31] is a generalization of Theorems 6.6 (ii) and 6.9 below, when μ is nonatomic. (Proof of the latter generalization, Theorem 7, can be simplified by referring to [13] as in the proof of Lemma 6.8 (ii) below.)

LEMMA 6.8. *Suppose $\Gamma \subset \mathscr{E}_{k+1}$ is convex, $K =$* **cl** Γ, *$\{a : (u, a) \in K$ for some $u\}$ is bounded above, and $\{u : (u, a) \in K$ for some $a\}$ is bounded. Then:*
(i) **ext upbdy** K = (**ext** K) $\cap$ **upbdy** K;
(ii) *if* **ext upbdy** $K \subset \Gamma$, *then Γ is upper closed;*
(iii) *if $\Gamma = \Gamma^1 + \Gamma^2$ (vector sum) and Γ^1 and Γ^2 are upper closed, so is Γ.*

Proof. Denying (i), suppose $(w, d) \in$ **ext upbdy** K and $(w, d) \notin$ **ext** K. Choose $0 < \alpha < 1, (w', d') \in K -$ **upbdy** K, and $(w'', d'') \in K$ such that $(w, d) = \alpha(w', d')$

$+ (1 - \alpha)(w'', d'')$. Pick $(w', b) \in K$ such that $b > d'$. Let $a = \alpha b + (1 - \alpha)d''$. Then $(w, a) \in K$ and $a > d$ contrary to $(w, d) \in \mathbf{upbdy}\, K$, so (i) follows.

To prove (ii), because of the boundedness hypotheses, it suffices to suppose $(w, d) \in \mathbf{upbdy}\, K$ and to show $(w, d) \in \Gamma$. Let A be the convex hull of $\mathbf{ext}\, K$ and $A' = \{0^k\} \times \{a : a \leqq 0\}$. By Theorem 6 of § 2.5 of [13], $K \subset A + A'$, since under the boundedness hypotheses, either A' or $\{0^{k+1}\}$ is the characteristic cone of K. We have $(w, d) = (u, a) + (0^k, a')$ with $(u, a) \in A$ and $a' \leqq 0$. Then $a' = 0$ else $(w, d) \notin \mathbf{upbdy}\, K$. Hence $(w, d) = \sum_{i=1}^m \alpha_i(w^i, d^i)$ with $\alpha^i > 0$ and $(w^i, d^i) \in \mathbf{ext}\, K$ for $i = 1, \cdots, m$ and $\sum_{i=1}^m \alpha^i = 1$. If for some j, $(w^j, d^j) \notin \mathbf{upbdy}\, K$, take $(w^j, d') \in K$ with $d' > d^j$, whence $(w, d + \alpha^j(d' - d^j)) \in K$, contrary to $(w, d) \in \mathbf{upbdy}\, K$. Thus for $i = 1, \cdots, m$, $(w^i, d^i) \in \mathbf{ext\, upbdy}\, K$ by (i), so $(w^i, d^i) \in \Gamma$. Hence, $(w, d) \in \Gamma$.

To prove (iii), suppose $(w, d) \in K$. For $n \in \omega$ and $i = 1, 2$, choose $(w^{ni}, d^{ni}) \in \Gamma^i$ such that $(w^{n1} + w^{n2}, d^{n1} + d^{n2}) \to (w, d)$. Because of the boundedness hypotheses and the fact that $d^{n1} + d^{n2} \to d \in \mathscr{E}_1$, for $i = 1, 2$, $\{(w^{ni}, d^{ni}) : n \in \omega\}$ is bounded; let it have $(w^i, d^i) \in \mathbf{cl}\, \Gamma^i$ as a limit point and choose $(w^i, b^i) \in \Gamma^i$ such that $b^i \geqq d^i$. Then $(w^1, d^1) + (w^2, d^2) = (w, d)$, so $(w, b^1 + b^2) \in \Gamma$ and $b^1 + b^2 \geqq d$. This completes the proof.

THEOREM 6.9. *Suppose μ is σ-finite,* **rng** *C is bounded,* **rng** *E is bounded above, and $Z(x)$ is closed for $x \in X$. Then* **fin rng** *(C, E) is upper closed.*

Proof. If μ is nonatomic, the theorem follows from Lemma 6.8 (i) and (ii) (with $\Gamma = \mathbf{fin\, rng}\, (C, E)$) and Theorems 3.1 and 6.6 (i) (alternatively apply Theorem 7 of Olech [31]). Using this fact, Theorem 6.3 (ii), and the partitioning of X in the proof of Theorem 6.6 (ii), one obtains the theorem from Lemma 6.8 (iii).

THEOREM 6.10. *Let μ be nonatomic and $p \in \Phi$ with $C(p)$ finite and $E(p) = \infty$. Then:*

(i) *if $w \in$* **fin rng** *C, $v(w) > -\infty$, and $0 < \alpha < 1$, then there exists an $s \in \Phi$ such that $E(s) = \infty$ and $[C(s) = \alpha w + (1 - \alpha)C(p)$ or $C(s) = (1 - \alpha)w + \alpha C(p)]$;*

(ii) *if $C(p) \in \Lambda \subset$* **fin rng** *C, Λ is a line segment, and $v(u) > -\infty$ for all $u \in \Lambda$, then for all $v \in$* **int** *Λ there exists a (strongly optimal) $p^* \in \Phi$ such that $C(p^*) = v$ and $E(p^*) = \infty$;*

(iii) *if $v(u) > -\infty$ for all $u \in$* **int rng** *C, then for all $v \in$* **int rng** *C, there exists a (strongly optimal) $p^* \in \Phi$ such that $C(p^*) = v$ and $E(p^*) = \infty$.*

Proof. By Theorem 3.1 and Remark 3.5, if $v(u) > -\infty$ for $u \in \mathbf{int\, rng}\, C$, then **int rng** C is convex, so (iii) follows from (ii).

Let the hypothesis of (i) hold. Choose $r \in \Phi$ such that $C(r) = w$ and $E(r) > -\infty$. Again applying Theorem 3.1 and Remark 3.5, we choose a measurable $S \subset X$ and $s, \bar{s} \in \Phi$ such that $s(x) = p(x)$ for $x \in S$, $s(x) = r(x)$ for $x \in X - S$, $\bar{s}(x) = r(x)$ for $x \in S$, $\bar{s}(x) = p(x)$ for $x \in X - S$, $C(s) = (1 - \alpha)w + \alpha C(p)$, and $C(\bar{s}) = \alpha w + (1 - \alpha)C(p)$. Then $E(s) = \infty$ or $E(\bar{s}) = \infty$, else $E(p) \neq \infty$ in contradiction. This proves (i).

Suppose the hypothesis of (ii) holds. Since $v \in \mathbf{int}\, \Lambda$, we may choose $j \in \omega$ sufficiently large that $v + 2^{-j}[v - C(p)] \in \Lambda$. By inductive application of (i) with $\alpha = \frac{1}{2}$, one finds q such that $E(q) = \infty$ and $C(q) = v - 2^{-j}[v - C(p)]$. An additional application of (i) yields the p^* desired in (ii). This completes the proof.

Remark 6.11. We show by example that in Theorems 6.6 and 6.9, we may not assume that $Z(x)$ is upper closed, instead of closed, for $x \in X$.

Let μ, k, X, Y, Ω, D, and c be as in Remark 5.5. Define

$$e(x,y)=\begin{cases}1-\frac{1}{2}|y| & \text{for } y\neq 0,\quad (x,y)\in\Omega,\\ 2 & \text{for } y=0,\quad x\in D,\\ 3 & \text{for } y=0,\quad x\in X-D.\end{cases}$$

Then for $x \in X$, $Z(x)$ is upper closed but not closed (the other hypotheses of Theorems 6.6 and 6.9 are satisfied). There exists no optimal $q^* \in \Phi$ such that $C(q^*) = 0$, although $0 \in \mathbf{rng}\, C$; hence, $\mathbf{rng}\,(C, E)$ is not upper closed. The example also shows that although Theorems 6.6 and 6.9 hold if for $x \in X$, $Y(x)$ is compact and $e(x, \cdot)$ and $c(x, \cdot)$ are continuous, they fail if $e(x, \cdot)$ is merely u.s.c.

If we change Theorem 6.9 by letting $Z(x)$ be upper closed instead of closed and by adding Condition (α), we obtain a valid statement, viz., Theorem 6.13 (ii) below. In the hypothesis of Theorem 6.13, the Borel conditions are on Ω, e, and c, while the upper closure condition is on Z. This adds complications to the proof in going from the Y, C, E structure to the Z, I structure and vice versa. As in Remark 5.4, a linear functional corollary is easily formed and proved whose hypothesis is in terms of Z and I. An obvious corollary is also formed by noting Remark 6.4. Both of these corollaries avoid the mixed structure nature of the hypothesis mentioned above, and both are weaker statements than Theorem 6.13.

LEMMA 6.12. *Suppose Condition* (α) *holds* (see § 5), *μ is nonatomic and σ-finite, and for $x \in X$, $\varnothing \neq Z^0(x) \subset Z(x)$, $Z^0(x)$ is upper closed, $\{(z_1, \cdots, z_k): z \in Z^0(x)\}$ is bounded, and $m(x) = \sup\{z_{k+1}: z \in Z^0(x)\}$. Suppose $\Delta \equiv \Omega \cap \{(x, y): (c(x, y), e(x, y)) \in Z^0(x)\}$ is a Borel set. Then there exists $g(x) \in Z^0(x)$ for $x \in X$ such that g is a measurable function and $[g_{k+1} = m$ or $\int_X g_{k+1}\, d\mu = \infty]$.*

Proof. To see that m is a measurable function, note that for $a \in \mathscr{E}_1$,

$$X \cap \{x: m(x) > a\} = \pi(\Delta \cap \{(x, y): e(x, y) > a\}),$$

which is a continuous image of a Borel set and thus measurable (§ 2.2.10 of [11]).

Let $P = \{x: m(x) = \infty\}$. Then P is measurable. Since μ is σ-finite and Borel regular, by Remark 4.2 we may assume m is a Borel function and P is a Borel set. Since μ is also nonatomic, we may choose disjoint Borel sets $P_1, P_2, \cdots$ such that $P = \bigcup_{n=1}^{\infty} P_n$ and such that if $\mu(P) > 0$, then $0 < \mu(P_n) < \infty$ for $n \in \omega$. Let $P_0 = X - P$,

$$\mathscr{A}_n = \Delta \cap \{(x, y): x \in P_n \text{ and } e(x, y)\mu(P_n) \geqq n\} \quad \text{for } n \in \omega,$$

$$\mathscr{A}_0 = \Delta \cap \{(x, y): x \in P_0 \text{ and } e(x, y) = m(x)\}.$$

Then, for $n = 0, 1, \cdots$, $\mathscr{A}_n$ is a Borel set, $\pi(\mathscr{A}_n) = P_n$, and we may apply Theorem 4.1 and Remark 4.2 to obtain a Borel function p_n on (almost all of) P_n such that $(x, p_n(x)) \in \mathscr{A}_n$ for $x \in P_n$. Let $g(x) = (c(x, p_n(x)), e(x, p_n(x)))$ for $x \in P_n$ and $n = 0, 1, \cdots$. Then g has the desired properties, which completes the proof.

THEOREM 6.13. *Suppose Condition* (α) *holds, μ is σ-finite,* $\mathbf{rng}\, C$ *is bounded, and for $x \in X$, $Z(x)$ is upper closed. Let Γ =* $\mathbf{fin\ rng}\, I$ (= $\mathbf{fin\ rng}\,(C, E)$). *Then*:

(i) *if μ is nonatomic and* $\mathbf{rng}\, E$ *is bounded above, then* $\mathbf{ext\ upbdy}\, \Gamma \subset \Gamma$;

(ii) *if* $\mathbf{rng}\, E$ *is bounded above, then Γ is upper closed*;

(iii) *if $N \subset \mathscr{E}_k$ is closed and convex, $N \cap$* $\mathbf{rng}\, C \neq \varnothing$, *and either μ is nonatomic or* $\mathbf{rng}\, E$ *is bounded above, then there exists $p^* \in \Phi$ such that $C(p^*) \in N$ and $E(p^*) = \max\{E(p): C(p) \in N\}$.*

Proof. Since **rng** C is bounded, using § 4 and Lemma 6.2 one argues as in proving Lemma 6.12 to show, if $\mathscr{F} \neq \emptyset$, $\{(z_1, \cdots, z_k): z \in Z(x)\}$ is bounded for $x \in X$.

To prove (i), we choose $b \in$ **ext upbdy** Γ. Noting Lemma 6.8 (i) and Remark 4.2, we obtain Borel functions $f^1, f^2, \cdots \in \mathscr{F}$ and f as given by Lemma 6.5.

For $x \in X$, define $Z^0(x) = Z(x) \cap \{z : z_i = f_i(x) \text{ for } i = 1, \cdots, k\}$, whence $Z^0(x)$ is upper closed and since $f(x) \in \mathbf{cl}\, Z(x)$, $\sup\{z_{k+1} : z \in Z^0(x)\} \geqq f_{k+1}(x)$. By Lemma 6.12, there exists $g(x) \in Z^0(x)$ for $x \in X$ such that $\int_X g_{k+1}\, d\mu \geqq \int_X f_{k+1}\, d\mu = b_{k+1}$. Since $g_i = f_i$ for $i = 1, \cdots, k$, $g \in \mathscr{F}$ and $I_i(g) = b_i$ for $i = 1, \cdots, k$. Also, $I_{k+1}(g) < \infty$, for **rng** E is bounded above. Since $b \in$ **upbdy** Γ, $I_{k+1}(g) = b_{k+1}$, i.e., $b \in \Gamma$, proving (i).

We prove (ii) by applying Theorems 3.1 and 6.3 (ii) and Lemma 6.8, as in the proof of Theorem 6.9, but using (i) instead of Theorem 6.6 (i).

If **rng** E is bounded above, then (iii) follows from (ii) and Lemma 6.1. To complete the proof, it suffices to show the following:

If μ is nonatomic, **rng** E *is not bounded above, $N \subset \mathscr{E}_k$ is closed and convex,*

(6.1) *and $N \cap$* **rng** $C \neq \emptyset$*, then there exists p^* such that*

$$C(p^*) \in N \text{ and } E(p^*) = \max\{E(p) : C(p) \in N\}.$$

We define $F = \mathbf{rng}\, C \cap \{w : v(w) > -\infty\}$. By Theorem 3.1 and Remark 3.5, F is convex. Hence, $N \cap F$ is convex. If $N \cap F = \emptyset$, (6.1) is trivial, so we assume

$$N \cap \mathbf{int}\, F \neq \emptyset \quad \text{or} \quad \emptyset \neq N \cap F \subset \mathbf{bdy}\, F. \tag{6.2}$$

Since $N \cap F$ is convex, if $N \cap F \subset \mathbf{bdy}\, F$, then $N \cap F$ is contained in a supporting hyperplane of F. Thus, we may choose $v^0 \in N \cap F$ and $\eta \in \mathscr{E}_k$ such that

$$\eta = 0^k \quad \text{and} \quad v^0 \in N \cap \mathbf{int}\, F, \qquad \text{if } N \cap \mathbf{int}\, F \neq \emptyset; \tag{6.3}$$

$$\eta \cdot v^0 \leqq \eta \cdot w \quad \text{for } w \in F \text{ with equality when } w \in N \cap F; \tag{6.4}$$

$$\eta_k \neq 0 \quad \text{if } N \cap F \subset \mathbf{bdy}\, F \quad \text{(reordering coordinates, if necessary)}. \tag{6.5}$$

Fix $p_0 \in \Phi$ such that $C(p_0) = v^0$, $E(p_0) > -\infty$, and $c(\cdot, p_0(\cdot))$ is a Borel function (see Remark 4.2). If $E(p_0) = \infty$, p_0 serves as p^*, so assume $|E(p_0)| < \infty$.

We claim that

$$\eta \cdot c(x, p_0(x)) \leqq \eta \cdot c(x, y) \quad \text{for } y \in Y(x), \quad x \in X. \tag{6.6}$$

If $\eta = 0^k$, then (6.6) holds trivially. If $\eta_k > 0$, let $\lambda = (1/\eta_k)(\eta_1, \cdots, \eta_{k-1}, 0)$, $C^0 = (C_1, \cdots, C_{k-1}, E)$, and $E^0 = -C_k$. By (6.4),

$$E^0(p_0) - \sum_{\lambda_i \neq 0} \lambda_i C_i^0(p_0) \geqq E^0(s) - \sum_{\lambda_i \neq 0} \lambda_i C_i^0(s) \quad \text{whenever } C(s) \in F.$$

By Theorem 2.2 ((iii) implies (i)), we find that the preceding inequality holds for all $s \in \Phi$. Theorem 5.1 yields that for $x \in X$, $p_0(x)$ maximizes $-c_k(x, \cdot) - \sum_{i=1}^{k-1} \lambda_i c_i(x, \cdot)$ over $T(x)$, whence (6.6) follows. We argue similarly if $\eta_k < 0$.

For $x \in X$, let $Z'(x) = Z(x) \cap \{z : \eta \cdot (z_1, \cdots, z_k) = \eta \cdot c(x, p_0(x))\}$; then $Z'(x)$ is upper closed and $(c(x, p_0(x)), e(x, p_0(x))) \in Z'(x)$. With m and g given by Lemma 6.12, we take $\hat{q} \in \Xi$ such that $g = (c(\cdot, \hat{q}(\cdot)), e(\cdot, \hat{q}(\cdot)))$. If $g_{k+1} = m$, then $e(\cdot, \hat{q}(\cdot))$

$\geqq e(\cdot, p_0(\cdot))$; in any event, $e(\cdot, \hat{q}(\cdot))$ is integrable. Since **rng** C is bounded, by Lemma 6.2, $c(\cdot, \hat{q}(\cdot))$ is integrable, i.e., $\hat{q} \in \Phi$. We have

$$\begin{aligned} &C(\hat{q}) \in F, \quad \eta \cdot c(\cdot, \hat{q}(\cdot)) = \eta \cdot c(\cdot, p_0(\cdot)), \quad \text{and} \\ (6.7) \quad &[E(\hat{q}) = \infty \text{ or } e(\cdot, \hat{q}(\cdot)) \geqq e(\cdot, q(\cdot)) \\ &\text{whenever } q \in \Phi \text{ with } \eta \cdot c(\cdot, q(\cdot)) = \eta \cdot c(\cdot, p_0(\cdot))]. \end{aligned}$$

If $N \cap \textbf{int}\, F \neq \emptyset$, then $\eta = 0^k$ by (6.3), so $E(\hat{q}) \geqq E(q)$ for $q \in \Phi$. Since **rng** E is not bounded above by hypothesis of (6.1), $E(\hat{q}) = \infty$. By Theorem 6.10 (ii), there exists $p \in \Phi$ such that $C(p) = v^0$ and $E(p) = \infty$; such p serves as p^*.

Hence by (6.2), we assume $\emptyset \neq N \cap F \subset \textbf{bdy}\, F$. We prove (6.1) by induction on k, the number of one-dimensional constraints.

If $k = 1$, $\eta = (\eta_1) \neq (0)$ and $N \cap F$ is a singleton. Since $\eta \cdot C(\hat{q}) = \eta \cdot v^0$, $C(\hat{q}) = v^0$. Suppose $q \in \Phi$, $E(q) > -\infty$, and $C(q) \in N$. Then $C(q) \in N \cap F$, so $C(q) = v^0$. By (6.6) and (6.7), $\eta \cdot c(\cdot, \hat{q}(\cdot)) \leqq \eta \cdot c(\cdot, q(\cdot))$, so since $\eta \cdot C(\hat{q}) = \eta \cdot C(q)$, equality holds in this inequality. Hence, $E(\hat{q}) \geqq E(q)$ by (6.7) and $\hat{q}$ serves as p^*. Thus, (6.1) holds if $k = 1$.

Suppose $k > 1$ and (6.1) holds when there are $k - 1$ one-dimensional constraints. Let

$$\Omega' = \Omega \cap \{(x, y) : \eta \cdot c(x, y) = \eta \cdot c(x, p_0(x))\},$$

$$\Phi' = \Phi \cap \{q : \eta \cdot c(\cdot, q(\cdot)) = \eta \cdot c(\cdot, p_0(\cdot))\},$$

$c' = C|\Omega'$, $e' = e|\Omega'$, $C' = C|\Phi'$, $E' = E|\Phi'$, and $N' = N \cap \{w : \eta \cdot w = \eta \cdot v_0\}$. The hypotheses of (6.1) are satisfied by the primed replacements (were E' bounded above, (ii) would yield p^*), and $C'(q) \in N'$ is expressible as $k - 1$ one-dimensional constraints. By the induction hypothesis, there exists $p^* \in \Phi'$ such that

$$C'(p^*) \in N' \quad \text{and} \quad E'(p^*) = \max\{E'(q) : q \in \Phi' \text{ and } C'(q) \in N'\}.$$

Suppose $q \in \Phi$, $E(q) > -\infty$, and $C(q) \in N$; then $C(q) \in N \cap F$, so by (6.4), $\eta \cdot C(q) = \eta \cdot v^0$; thus, by (6.6), $\eta \cdot c(\cdot, q(\cdot)) = \eta \cdot c(\cdot, p_0(\cdot))$. Hence $q \in \Phi'$. Thus, $E(p^*) = \max\{E(q) : C(q) \in N\}$, proving (6.1) and hence the theorem.

Remark 6.14. We show by two examples, with a variation on each, that Theorem 6.13 and other results of this section cannot be extended in certain ways.

First, let $X = [0, \infty)$, μ be Lebesgue measure, $k = 1$, and for $x \in X$, $Y(x) = \{0, 1\}$, $e(x, 0) = c(x, 0) = 0$, $e(x, 1) = 1$, and $c(x, 1) = e^{-x}$. Then **rng** $C = [0, 1]$, $v(0) = 0$, $v(v) = \infty$ for $0 < v \leqq 1$, and for $0 \leqq v \leqq 1$, there exists $q \in \Phi$ such that $C(q) = v$ and $E(q) = v(v)$. Hence in Theorem 6.10 (iii) we cannot change $v \in \textbf{int rng}\, C$ to $v \in \textbf{rng}\, C$, since the conclusion fails with $v = 0$. Also, if **rng** E is not bounded above, then v need not be u.s.c., even if optimal functions exist for all costs, **rng** C is compact, and **fin rng** (C, E) is convex.

If we redefine $e(x, 0) = -1$ for $x \in X$, we have $v(0) = -\infty$ and $v(v) = \infty$ for $0 < v \leqq 1$.

For the second example, let $k = 1$, $X = \omega$, and $Y(x) = \{0, 1, 2\}$ and $\mu(\{x\}) = 1$ for $x \in X$. Define c by $c(x, 0) = 0$, $c(x, 1) = 3 \cdot 4^{-x}$, and $c(x, 2) = 4^{1-x}$ for $x \in X$. For $n \in \omega \cup \{\infty\}$, define $q_n \in \Phi$ by, for $x \in X$, $q_n(x) = 1$ if $x < n$, $q_n(x) = 2$ if $x = n$, and $q_n(x) = 0$ if $x > n$. Since $\int_{x \leqq a} c(x, 1)\, d\mu x = 1 - c(a + 1, 2)$ for $a \in \omega$, one may show that for $q \in \Phi$, $C(q) = 1$ iff $q = q_n$ for some $n \in \omega \cup \{\infty\}$.

Define $e(x, 0) = 0$, $e(x, 1) = -1$, and $e(x, 2) = x - 2^{-x}$ for $x \in X$. Then for $n \in \omega$, $E(q_n) = 1 - 2^{-n}$ and $E(q_\infty) = -\infty$. Thus, $v(1) = 1$, but $E(q) < 1$ whenever $C(q) = 1$.

If we alternatively define $e(x, 2) = 2x$ for $x \in X$, then $E(q_n) = n + 1$ for $n \in \omega$ and again $E(q_\infty) = -\infty$, whence $v(1) = \infty$, but $E(q) < \infty$ whenever $C(q) = 1$.

Now consider the existence corollaries to Theorems 6.3(ii), 6.9, and 6.13 (ii) formed from Lemma 6.1 with $N = \{v\}$. The example and its alternative show (with $v = 1$) that we cannot substitute either the hypothesis "$v(v) < \infty$" or the hypothesis "$v(v) = \infty$" for the hypothesis "**rng** E is bounded above" in any of these corollaries, nor may we substitute either for the hypothesis "μ is nonatomic or **rng** E is bounded above" appearing in Theorem 6.13 (iii).

The alternative definition of $e(\cdot, 2)$ above also shows that we may not delete the assumption that μ is nonatomic in Theorem 6.10. One can, however, easily show the following: if μ is purely atomic and σ-finite, $E(r) = \infty$ for some $r \in \Phi$, and **rng** C is bounded, then $\{C(p): E(p) = \infty\}$ is dense in **rng** C.

Acknowledgments. We wish to acknowledge some conversations with Professors Hubert Halkin and L. W. Neustadt and a referee's comments, which were extremely helpful in acquainting us with prior literature. As noted in § 4, the new proof of von Neumann's selection theorem given here is due to Professor Herbert Federer and a variation on this proof was provided by Professor J. C. Oxtoby, both communicated privately.

Added in proof. An extension of Theorem 6.13 (ii) above and Theorem 7 of [31] is announced in [44].

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