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OPTIMAL SEARCH USING UNINTERRUPTED CONTACT INVESTIGATION*

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Abstract. This paper finds the optimal search plan (in the sense of minimizing mean time to find the target) for a class of searches for a stationary target. In this class the target must be contacted by one sensor and identified by another. Complicating the search is the possibility of false targets which must be identified to be distinguished from the target. Search plans are described by a search effort density function and a policy for investigating contacts. The existence of false targets is determined by a known density function for the mean number of false targets in a region.

Under the condition that contact investigation, once begun, must not be interrupted until the contact is identified, it is shown that the optimal plan, in a specified class, is to allocate search effort according to a Neyman–Pearson type of allocation and to investigate contacts immediately. It is also shown that if at any time an optimal search is stopped and it is decided to replan the search, the optimal plan is to continue the original plan. This last result makes use of the posterior target location distribution when there are unidentified contacts. This distribution is also found in this paper. Examples of optimal search plans are presented as well as an example to show that, in general, it is not possible to maximize the probability of finding the target at each instant of time during a search.

Introduction. The problem of optimal allocation of search effort has received considerable attention in the case where there are no false targets (i.e., the search sensor has perfect discrimination). The bibliography of search literature given by Dobbie in [4] lists numerous papers on this subject. An additional paper on this subject is that of Arkin [1]. A classical paper on this problem is Koopman's [9]. However, we are concerned with the case where search takes place in two phases. The first phase, broad search, scans the area with a sensor which is capable of detecting the target but cannot distinguish, from the target, objects which give responses similar to the target. Thus, there may be false targets. A sensor response which might be caused by the target is called a contact. Once a contact is generated, it is necessary to go the second phase, contact investigation, to decide whether the contact was caused by a false target or not. It is assumed that contact investigation is capable of deciding whether or not the contact was produced by a false target after a finite, but possibly random, amount of investigation.

In § 1, the search model used in this paper is outlined and in § 2 and § 3, we find search plans which are optimal in the sense of minimizing the mean time to find the target. In § 4, we show that the optimal plan is additive in the sense that if at any time during the search it is decided to replan the search to minimize the mean time remaining to find the target, the optimal plan is to continue searching according to the original optimal plan. The proof of the additivity result makes use of the a posteriori target location distribution when there are contacts that have not been investigated. The derivation of this distribution is given in the last section.

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One would like to find plans which maximize $P(t)$, the probability of finding the target by time t for all t, but we show by example in § 5 that in general, it is not possible to do this in the presence of false targets. In § 6, examples of optimal plans are presented.

For a class of searches not involving false targets in which one may maximize $P(t)$ for all t, Dobbie [3] has shown that the optimal plan is additive in the sense that it maximizes the increase in probability resulting from each increment of search effort.

1. Search model. We consider search for a target whose a priori location distribution has a density function f defined on a region R of Euclidean n-space. The search is complicated by the fact that the sensor used to detect the target does not have perfect discrimination. That is, the sensor may detect objects which cannot be distinguished from the target without further investigation, presumably with a different sensor. Any such object which is detected is called a *contact* and any such object which is not the target is called a *false target.*

The search takes place in two phases. The broad search phase is conducted using a sensor which can detect the target but not positively identify it. In order to investigate a contact, the broad search must stop and a contact investigation must begin. Once a contact investigation has begun, it must continue until the contact is identified. This is called *uninterrupted contact investigation.* At the end of a random time, the contact is correctly identified either as being or not being the target. We assume that investigation of one contact makes no contribution to investigation of any other contact or to the broad search. Also once a contact has been investigated it is, in effect, eliminated and will not be classified as a contact if detected again.

We distinguish between two types of search time. Cumulative broad search time will be denoted by s. Cumulative time spent in all aspects of search and investigation will be denoted by t. To avoid confusion, we say a target (real or false) is *contacted* when it appears as a contact, and that it has been *identified* when contact investigation shows that contact to be a real or false target. We say the target has been *found* when it has been contacted and identified.

For the broad search, it is assumed that there is a *local effectiveness function* b defined on $[0, \infty)$ such that:

(i) $0 \leqq b \leqq 1$, $b(0) = 0$ and $\lim_{z \to \infty} b(z) = 1$,

(ii) b', the derivative of b, exists, is strictly positive, continuous, and strictly decreasing.

A broad search is described by the *broad search density function* m which is a nonnegative Borel measurable function defined on $R \times [0, \infty)$ such that:

(iii) $m(x, \cdot)$ is nondecreasing for each x in R,

(iv) $\int_R m(x, s)\, dx = Us$ for all $s \geqq 0$,

where U is a fixed positive constant called the *broad search rate.* The class of such functions m is called M. It is assumed that for each m in M,

$$P(m, s) = \int_R f(x) b(m(x, s))\, dx$$

is the probability of contacting the target by broad search time s using the allocation of effort given by m. Thus $b(m(x, s))$ is the conditional probability of contacting the target by broad search time s, given that the target is located at x.

The above modeling of the broad search detection process is essentially the same as the search model given by DeGuenin in [2] which is an extension of Koopman's model in [9]. The characterization of a search plan by a nondecreasing effort density function is similar to Arkin's characterization in [1]. Moreover, Arkin considers searches (without false targets) in which b may be any probability distribution function such that $b(0) = 0$. Both Arkin and DeGuenin allow the function b to depend on x.

For false targets we assume that there is a nonnegative Borel measurable function δ defined on R such that for any Borel set B in R,

$$\varphi(B, s) \equiv \int_B \delta(x)b(m(x, s))\, dx$$

is the expected number of false targets contacted in B by broad search time s. The definition of φ given above reflects the fact that we are viewing false targets as real objects with the same detection characteristics as the target. The form of φ tacitly assumes that once a false contact has been contacted, it can be recorded or marked in such a way that if the same false target is contacted more than once, the contacts will be recognized as coming from the same object.

The forms of P and φ are identical except for the functions δ and f. If there were an unknown number of targets and if f were the density function for the mean number of targets in a region, then $P(m, s)$ would be the expected number of targets found by broad search time s and the symmetry between φ and P would be complete. In this paper, however, we assume that there is exactly one target.

In order to calculate the mean amount of time spent identifying false targets and in order to find the posterior target location distribution when there are unidentified contacts, one must give a more detailed probabilistic structure for the false target process. Thus we define, for any Borel subset B of R and $s \geqq 0$, the random variable

$$\Delta(B, s) \equiv \left[\begin{array}{c}\text{the number of false targets contacted} \\ \text{in } B \text{ by broad search time } s\end{array}\right]$$

and assume that

$$\Pr\{\Delta(B, s) = n\} = \frac{[\varphi(B, s)]^n e^{-\varphi(B,s)}}{n!} \quad \text{for } n = 0, 1, \cdots. \tag{1.1}$$

To motivate this assumption we consider conditions (a) through (d) below. These conditions are essentially the same as those given in [8, p. 337], where it is shown that (1.1) follows from the conditions given below along with the definition of φ as the mean number of false targets contacted:

(a) $0 < \Pr\{\Delta(B, s) = 0\} < 1$ if $\varphi(B, s) > 0$;

(b) the probability distribution of $\Delta(B, s)$ depends only on $\varphi(B, s)$ and

$$\Pr\{\Delta(B, s) \geqq 1\} \to 0 \quad \text{as } \varphi(B, s) \to 0;$$

(c) if $B_1, \cdots, B_n$ $(n \geqq 1)$ are disjoint regions, then $\Delta(B_1, s), \cdots, \Delta(B_n, s)$ are mutually independent random variables;

(d) $$\lim_{\varphi(B,s)\to 0} \frac{\Pr\{\Delta(B,s) \geqq 1\}}{\Pr\{\Delta(B,s) = 1\}} = 1.$$

Assumption (a) states that if some broad search has been performed in an area where $\delta > 0$, then there is positive (but not equal to one) probability of contacting no false targets. The first part of (b) says essentially that the number of false contacts does not depend on the shape of the region but only on the area of the region, the false target density, and the thoroughness of the search. The motivation for the second part of (b) is obvious after one notes that $\varphi(B, s)$ approaches zero whenever the area of B approaches 0, the false contact density in B approaches 0, or the search effort density function approaches 0 in B. The assumption in (c) says that the contacting of false targets in one area does not affect the contacting of them in any disjoint area. This assumption requires, for example, that the operators of the broad search sensor do not improve their ability to distinguish false targets from the real target as the search progresses. Assumption (d) states that as $\varphi(B, s) \to 0$, the probability of contacting more than one false target becomes small compared to the probability of contacting exactly one.

We further assume that the contacting of false targets is stochastically independent of contacting the target.

For the mean time to identify contacts we suppose that there is a nonnegative Borel measurable function T defined on R such that $T(x)$ is the mean time to identify a false target contacted at x. Note that we assume T depends only on the location at which the contact is made and not on δ, b, or the time at which the contact is made. The proof of Theorem 1.2 in [8, p. 343] may be used with trivial modifications to show that if $\Delta(R, s) = k$, then the distribution of the location of the k false targets contacted is that of k independent draws from a distribution with density $\delta(x)b(m(x, s))/\varphi(R, s)$. It follows that

$$e^{-\varphi(R,s)} \sum_{k=1}^{\infty} \frac{[\varphi(R,s)]^k}{k!} k \int_R \frac{T(x)\delta(x)b(m(x,s))}{\varphi(R,s)}\,dx = \int_R T(x)\delta(x)b(m(x,s))\,dx \tag{1.2}$$

is the expected amount of time to identify all of the *false* targets that have appeared as contacts by broad search time s using the allocation m.

We shall also assume that the mean time to identify the contact which is the target depends only on the location of that contact.

The above assumptions could be easily relaxed in several ways. One could allow b to depend on x or replace the assumption in (ii) that b' be strictly decreasing by the assumption that b' be nonincreasing. Also one could allow for a different local effectiveness function against false targets than against the target. It would still be possible to specify a Neyman–Pearson allocation for the optimal search plan but in a manner which is far less explicit than that given below. Also one could relax the restriction that b' be strictly positive in the manner described after the proof of Theorem 2.

We shall write m_s for $m(\cdot, s)$, and for any Lebesgue integrable g, we use v to denote Lebesgue measure and write $\int_R g\,dv$ for $\int_R g(x)\,dx$ whenever convenient.

2. Optimal broad search distribution for immediate identification plan. In this section we find the allocation m^* of broad search effort which minimizes the mean time to find the target among all plans which use uninterrupted immediate contact investigation. Throughout this paper we restrict ourselves to plans which produce probability 1 of finding the target as time approaches ∞.

We shall find m^* by using the nonlinear functional version of the Neyman–Pearson lemma which appears as Theorem 1 in [10]. For the convenience of the reader we quote a form of that theorem which is suited to the purposes of this paper.

THEOREM 1. *Let e and c be real-valued functions defined on $R \times [0, \infty)$. For $x \in R$, let $e'(x, \cdot)$ and $c'(x, \cdot)$ denote the derivative of $e(x, \cdot)$ and $c(x, \cdot)$ respectively. Assume that for each x in R, $e'(x, \cdot)$ and $c'(x, \cdot)$ exist and are Riemann integrable on bounded intervals of $[0, \infty)$. Let G be the nonnegative Borel measurable functions, defined on R, such that*

$$E(g) \equiv \int_R e(x, g(x))\,dx \quad \textit{and} \quad C(g) \equiv \int_R c(x, g(x))\,dx$$

exist and are finite.

Suppose that there exist a $\Lambda > 0$ and $g^* \in G$ such that

$$e'(x, z) \geqq \Lambda c'(x, z) \quad \textit{whenever } 0 < z < g^*(x), \tag{2.1}$$

$$e'(x, z) \leqq \Lambda c'(x, z) \quad \textit{whenever } g^* < z < \infty. \tag{2.2}$$

Then for any $g \in G$,

$$E(g) \geqq E(g^*) \quad \textit{implies} \quad C(g) \geqq C(g^*).$$

Thinking of E and C as effectiveness and cost respectively, we see that g^* is a function which gives the most effectiveness for the cost.

Define

$$e(x, z) = f(x)b(z), \tag{2.3}$$

$$c(x, z) = z/U + T(x)\delta(x)b(z) \qquad \text{for } x \text{ in } R, \quad z \geqq 0. \tag{2.4}$$

Note that $E(m_s) = P(m, s)$. Since

$$s = \frac{1}{U}\int_R m_s\,dv,$$

$C(m_s)$ gives the expected amount of time spent in broad search and in investigating the false targets contacted by broad search time s.

Thus,

$$\mu(m) \equiv \int_0^\infty C(m_s)P(m, ds)$$

is the mean time spent in broad search and false contact investigation before the investigation of the contact which is the target begins. Since the mean time to investigate the contact which is the target depends only on the location of that contact, minimizing μ is equivalent to minimizing the mean time to find the target.

In order to minimize μ, we shall find an $m^* \in M$ such that at each broad search time s there is a $\lambda_s > 0$ satisfying

$$e'(x, z) \geqq \lambda_s c'(x, z) \quad \text{whenever } 0 < z < m^*(x, s), \tag{2.5}$$

$$e'(x, z) \leqq \lambda_s c'(x, z) \quad \text{whenever } m^*(x, s) < z < \infty. \tag{2.6}$$

By Theorem 1, such an m^* has the property that it minimizes the expected time to reach any fixed probability of finding the target. It will be shown in Theorem 2 that this property guarantees that m^* minimizes μ.

We approach the problem of finding m^* by first finding the optimal broad search allocation as a function of the Λ which appears in (2.1) and (2.2). For the moment let us fix Λ and try to solve (2.1) and (2.2) for g^*. Note that by the assumptions (i), (ii) in § 1,

$$\frac{e'(x, \cdot)}{c'(x, \cdot)} = \frac{f(x)b'(\cdot)}{1/U + T(x)\delta(x)b'(\cdot)} \tag{2.7}$$

is strictly decreasing and continuous. Thus if there is a $g^*(x) \geqq 0$ satisfying

$$e'(x, g^*(x))/c'(x, g^*(x)) = \Lambda, \tag{2.8}$$

it is unique. If we define

$$r(x, \Lambda) = \frac{\Lambda}{U(f(x) - \Lambda\delta(x)T(x))} \qquad \text{for } x \text{ in } R, \quad \Lambda > 0,$$

then one may check by substituting (2.7) into (2.8) that the solution to (2.8) is given by

$$g^*(x) = b'^{-1}(r(x, \Lambda))$$

provided that

$$b'(0) \geqq r(x, \Lambda) > 0. \tag{2.9}$$

The decreasing nature of the ratio in (2.7) guarantees that (2.1) and (2.2) are satisfied. If Λ is so large that (2.9) is not satisfied, then $g^*(x) = 0$ satisfies (2.1) and (2.2).

In view of the above discussion, we define

$$d(x, \Lambda) = \begin{cases} b'^{-1}(r(x, \Lambda)) & \text{if } b'(0) \geqq r(x, \Lambda) > 0, \\ 0 & \text{otherwise} \end{cases} \tag{2.10}$$

for x in R and $\Lambda > 0$. Then $g^*(x) = d(x, \Lambda)$ satisfies (2.1) and (2.2) for all x in R. Each choice of Λ corresponds to a fixed amount of search effort $A(\Lambda)$ as follows:

$$A(\Lambda) \equiv \begin{cases} \int_R d(x, \Lambda)\, dx & \text{for } \infty > \Lambda > 0, \\ \infty & \text{for } \Lambda = 0, \\ 0 & \text{for } \Lambda = \infty. \end{cases} \tag{2.11}$$

Thus to define $m^*(x, s)$, one need only find λ_s such that $A(\lambda_s) = Us$ for each $s > 0$. In Theorem 2 below it is shown that by suitably restricting the domain of A, one may define A^{-1} such that

$$Us = A(A^{-1}(Us)) \qquad \text{for } s > 0 \tag{2.12}$$

and that m^* defined by

$$m^*(x, s) = d(x, \lambda_s) \qquad \text{for } x \in R \text{ and } s > 0, \tag{2.13}$$

where

$$\lambda_s = A^{-1}(Us) \qquad \text{for } s > 0 \tag{2.14}$$

satisfies (2.5) and (2.6) for all $x \in R$ and $s > 0$. Moreover, Theorem 2 shows that

$$\mu(m^*) \leqq \mu(m) \qquad \text{for all } m \in M. \tag{2.15}$$

By the discussion above, (2.15) is equivalent to saying that m^* is the broad search density which minimizes the mean time to find the target among all plans which use uninterrupted contact investigation and investigate contacts immediately.

THEOREM 2. *Let*

$$\Lambda_l = \sup \{\Lambda : A(\Lambda) = \infty\}, \tag{2.16}$$

$$\Lambda_u = \inf \{\Lambda : A(\Lambda) = 0\}. \tag{2.17}$$

By restricting the domain of A to (Λ_l, Λ_u), we may define an inverse function A^{-1} for A such that for all $s > 0$,

$$m^*(x, s) = d(x, A^{-1}(Us)) \tag{2.18}$$

has the property that

$$E(g) \geqq E(m_s^*) \quad \textit{implies} \quad C(g) \geqq C(m_s^*) \quad \textit{for any } g \in G \textit{ and } s > 0, \tag{2.19}$$

that

$$\mu(m^*) \leqq \mu(m) \qquad \textit{for any } m \in M, \tag{2.20}$$

and that m^* minimizes the mean time to find the target among all $m \in M$ when contacts are investigated immediately using uninterrupted contact investigation.

Proof. We shall prove the theorem by showing that A^{-1} exists and that for $\lambda_s = A^{-1}(Us)$, m^* satisfies (2.5) and (2.6) for all $x \in R$ and $s > 0$. The implication in (2.19) then follows from Theorem 1. We then use (2.19) to prove (2.20).

First, we prove that A^{-1} exists and is defined on $(0, \infty)$. By virtue of assumptions (i) and (ii), we have for $z \geqq 0$,

$$1 \geqq \int_0^z b'(y)\, dy \geqq zb'(z).$$

Thus $b'(z) \leqq 1/z$ which implies that $b'^{-1}(z) \leqq 1/z$. So for $\infty > \Lambda > 0$,

$$A(\Lambda) = \int_R d(x, \Lambda)\, dx \leqq \int_{[r(x,\Lambda)>0]} \frac{dx}{r(x, \Lambda)} \leqq \int_R \frac{Uf(x)\, dx}{\Lambda} \leqq \frac{U}{\Lambda}.$$

Hence

$$\lim_{\Lambda \to \infty} A(\Lambda) = 0. \tag{2.21}$$

Since we have assumed $b' > 0$,

$$b'^{-1}(r(x, \Lambda)) \uparrow \infty \quad \text{as } \Lambda \downarrow 0 \qquad \text{for each } x \text{ in } R,$$

and we have by the monotone convergence theorem that

$$\lim_{\Lambda \to 0} A(\Lambda) = \infty. \tag{2.22}$$

Because b'^{-1} is continuous and decreasing, we may use the monotone convergence theorem to show that A is continuous. Since b'^{-1} is strictly decreasing, A is strictly decreasing on (Λ_l, Λ_u), and we may define A^{-1} to be the function with domain $(0, \infty)$ and range (Λ_l, Λ_u) such that for each $z > 0$,

$$A(A^{-1}(z)) = z.$$

By virtue of (2.21) and (2.22), and the fact that A is continuous and strictly decreasing on (Λ_l, Λ_u), A^{-1} is well-defined.

Following Example 3 of [10], we observe that for $\lambda_s > 0$,

$$f(x)b'(m^*(x, s)) = \lambda_s(1/U + T(x)\delta(x)b'(m^*(x, s)))$$

holds if and only if

$$m^*(x, s) = d(x, \lambda_s) \quad \text{and} \quad b'(0) \geqq r(x, \lambda_s) > 0.$$

Thus by the decreasing nature of b', (2.5) and (2.6) hold for $m^*(x, s) = d(x, \lambda_s)$, where $\lambda_s = A^{-1}(Us)$. The implication in (2.19) now follows from Theorem 1.

In order to prove (2.20), we define

$$C_+(m, p) = \inf\{C(m_s) : s \geqq 0 \text{ and } P(m, s) \geqq p\} \qquad \text{for } 0 \leqq p < 1 \text{ and } m \text{ in } M.$$

Fix p. Since b is continuous and strictly increasing, one may verify that $P(m^*, \cdot)$ is continuous and strictly increasing so that there is a unique number α such that $P(m^*, \alpha) = p$. Thus, $C_+(m^*, p) = C(m_\alpha^*)$. We claim that

$$C(m_\alpha^*) = C_+(m^*, p) \leqq C_+(m, p) \qquad \text{for all } m \text{ in } M. \tag{2.23}$$

To see this, we suppose $C_+(m^{**}, p) < C_+(m^*, p)$ for some m^{**} in M. Then by the definition of C_+, there must exist an $s \geqq 0$ such that

$$E(m_s^{**}) = P(m^{**}, s) \geqq p \quad \text{and} \quad C(m_s^{**}) < C(m_\alpha^*).$$

But then $g = m_s^{**}$ would violate (2.19).

By Lemma 1, proved below, it follows that

$$\mu(m) = \int_0^1 C_+(m, p)\, dp, \tag{2.24}$$

and from (2.23) and (2.24), it follows that m^* satisfies (2.20) and minimizes the mean time to find the target. This proves the theorem.

The assumption that $b' > 0$ made in (ii) is used to guarantee that $\lim_{\Lambda \to 0} A(\Lambda) = \infty$. One may relax this assumption to $b' \geqq 0$ by dealing separately with the cases where $A(\Lambda)$ has a finite and an infinite limit respectively as Λ approaches 0. If the limit is infinite, the argument in Theorem 1 remains the same. If the limit is finite, then after a finite amount of search nothing is gained by adding more search. In this case the search stops after a finite time s_0 of broad search, and the optimal allocation of broad search effort for $0 < s \leqq s_0$ is given by (2.18).

LEMMA 1. *Let D be a nondecreasing function defined on* $[0, \infty)$, *and let L be the distribution function of a probability measure on* $[0, \infty)$. *Define*

$$D_+(p) = \inf\{D(s) : s \geqq 0 \text{ and } L(s) \geqq p\} \qquad \text{for } 0 \leqq p < 1.$$

Then

$$\int_0^\infty D(s)L(ds) = \int_0^1 D_+(p)\,dp. \tag{2.25}$$

Proof. By Theorem 1 in [6, p. 19], we have

$$\int_0^\infty D_+(L(s))L(ds) = \int_0^1 D_+(p)\,dp.$$

Since L and D are nondecreasing,

$$D_+(L(s)) = \inf\{D(r) : r \geqq s\} = D(s).$$

Thus, (2.25) holds and the lemma is proved.

The techniques used to prove Theorem 2 may be used with only trivial changes to prove Theorem 2′ below. Theorem 2′ is of interest when one is trying to find explicit Neyman–Pearson type solutions to constrained extremal problems. Versions of this theorem have been given by Karlin [7] for the case where $R = [0, \infty)$, f is positive continuous and decreasing on R, and the second derivative of b is negative and continuous, and by Wagner [10] who was able to remove the assumption that the second derivative of b be continuous and to give an easier proof.

THEOREM 2′. *Let f be a nonnegative integrable function defined on R, a subset of n-dimensional Euclidean space. Let b, defined on* $[0, \infty)$, *be nonnegative, bounded and such that* $b(0) = 0$. *Moreover, let b' be continuous strictly positive, and strictly decreasing. Then for any* $Q > 0$,

$$g^*(\cdot) = d(\cdot, A^{-1}(Q))$$

maximizes

$$\int_R f b(g)\,dv$$

(*where $b(g)$ means b composed with g*) *among all nonnegative Borel functions g defined on R for which*

$$\int_R g\,dv \leqq Q,$$

where d is defined by (2.10), *A by* (2.11),

$$r(x, \Lambda) = \Lambda / f(x) \qquad \text{for } x \text{ in } R,$$

and A^{-1} *is the function whose domain is* $(0, \infty)$ *and range is* (Λ_l, Λ_u) *such that*

$$A(A^{-1}(z)) = z \qquad \text{for } z > 0,$$

where Λ_l *and* Λ_u *are given by* (2.16) *and* (2.17) *respectively.*

3. Immediate contact investigation with broad search allocation m^* is optimal. Here we show that the plan of immediate contact investigation with broad search density m^* is optimal in a much larger class of plans than heretofore considered. The class of plans considered in this section may be described as follows: A member Z of this class of plans prescribes when contacts are investigated and how broad search effort should be allocated as a function of the broad search time s, the locations and times at which contacts are made, and the times at which contacts are investigated. An investigation of a contact, once begun, is carried to completion. Because the occurrence of false contacts is a random phenomenon, $\mathbf{m}(\cdot, s, Z)$, the search density at time s, will in general be stochastic for a fixed search plan Z. It is assumed that each realization of $\mathbf{m}$ will be a broad search density function as defined in the first section.

Let G be the set of nonnegative Borel functions defined on R, and let G have the topology induced by uniform convergence (i.e., the topology induced by the supremum norm). We take the Borel subsets of G to be the measurable subsets of G.

In order to apply Neyman–Pearson optimization techniques, we define the class of plans Ξ. A plan Z is contained in Ξ if the following hold:

(A) Each realization of $\mathbf{m}(\cdot, \cdot, Z)$ is a broad search density function.

(B) For each $s > 0$ and Borel subset Γ of G,

$$F(\Gamma, s, Z) \equiv \Pr\{\mathbf{m}(\cdot, s, Z) \in \Gamma\}$$

is well-defined, and $F(\cdot, s, Z)$ defines a probability measure on Borel subsets of G.

(C) Contact investigation is uninterrupted in plan Z.

(D) Using plan Z, the probability of beginning the investigation of (and therefore by (C) identifying) a contact at x by broad search time s depends only on broad search time s and the location x and not on the number or location of other contacts or on the broad search density at x. For $s \geqq 0$ and $x \in R$, we let $X(x, s, Z)$ denote the probability, using plan Z, of identifying a contact at x by broad search time s.

Note that the plan Z^* which uses the deterministic broad search density m^* and identifies contacts immediately (i.e., $X(s, x, Z^*) = 1$ for all $x \in R$ and $s \geqq 0$) is in Ξ. Another plan $Z \in \Xi$ is the one which uses a deterministic broad search density m and defers contact identification until a fixed probability p of target detection is achieved. Since this probability is achieved at a fixed broad search time s_p, this plan corresponds to $X(x, s, Z) = 0$ for $s < s_p$ and $X(x, s, Z) = 1$ for $s \geqq s_p$ and all $x \in R$.

Although we shall prove that Z^* is optimal in Ξ, the following observation is useful for extending the class of plans in which Z^* is optimal. If Z is a plan (not necessarily in Ξ), then let $\tilde{\mu}(Z)$ be the mean time to find the target using this plan. If there is a sequence of plans $Z_n \in \Xi$ for $n = 1, 2, \cdots$, such that $\lim_{n\to\infty} \tilde{\mu}(Z_n) = \tilde{\mu}(Z)$, then $\tilde{\mu}(Z) \geqq \tilde{\mu}(Z^*)$. An open problem is to determine the class Ξ^* of plans which may be approximated (in the sense of mean time to find the target) by plans in Ξ, and thus extend the class of plans in which Z^* is optimal.

The probability of having found the target by broad search time s when using a plan $Z \in \Xi$ is

$$\mathbf{P}(Z, s) \equiv \int_G \int_R f X_s(Z) b(g)\, dv F(dg, s, Z), \tag{3.1}$$

where $X_s(Z) = X(\cdot, s, Z)$. For $g \in G$, let

$$J(g) = \int_R f X_s(Z) b(g)\, dv.$$

Since b is continuous on $[0, \infty)$, one may use the dominated convergence theorem to verify that J is a continuous function on G. Thus, the integral in (3.1) is well-defined. For the plans we consider, we shall require that

$$\mathbf{P}(Z, s) \to 1 \quad \text{as} \quad s \to \infty.$$

This is no restriction since the mean time to find the target is infinite if the above limit does not hold.

An argument similar to the one which led to (1.2) shows that the expected time spent investigating false targets contacted by broad search time s is

$$\int_G \int_R T\delta X_s(Z) b(g)\, dv F(dg, s, Z).$$

The above integral includes all time spent investigating false targets before any broad search is exerted beyond that expended by broad search time s.

For the remainder of this section we shall consider a special subclass Ξ' of Ξ in which for any given amount of broad search time, at most one contact is investigated before additional broad search effort is exerted. We note that for any plan in Ξ which investigates more than one contact without an intervening period of broad search, we can construct a corresponding plan in Ξ' by inserting arbitrarily small periods of broad search between contact investigations. Thus for any plan in Ξ we can find a plan in Ξ' whose mean time to find the target is arbitrarily close to that of the plan in Ξ. Thus the plan which is optimal (i.e., minimizes the mean time to find the target) in Ξ' is optimal in Ξ.

Note that

$$s = \int_G \int_R \frac{g}{U}\, dv F(dg, s, Z).$$

Thus $\mathbf{C}(Z, s)$, the expected amount of time spent in broad search and investigating false targets for a search in Ξ' which accumulates broad search time s before identifying the target, is given by

$$\mathbf{C}(Z, s) = \int_G \int_R \left(\frac{g}{U} + \delta T b(g) X_s(Z)\right) dv F(dg, s, Z).$$

For plans in Ξ' we can calculate μ, the mean duration of the search up to the time the investigation of the contact which is the target begins, by

$$\mu(Z) = \int_0^\infty \mathbf{C}(Z, s) \mathbf{P}(Z, ds).$$

Once again, we shall minimize the mean time to find the target by minimizing μ. Define

$$\mathbf{C}_+(Z, p) = \inf\{\mathbf{C}(Z, s) : s \geqq 0 \text{ and } \mathbf{P}(Z, s) \geqq p\}.$$

Then by Lemma 1,

$$\mu(Z) = \int_0^1 \mathbf{C}_+(Z, p)\, dp. \tag{3.2}$$

Let Z^* be the plan given by immediate investigation and the deterministic broad search density $\mathbf{m} = m^*$, with m^* as defined by (2.18). We shall show that Z^* is optimal in Ξ' in the sense of minimizing expected time needed to contact and identify the target. This will follow from (3.2) upon showing that for any Z in Ξ',

$$\mathbf{C}_+(Z, p) \geqq \mathbf{C}_+(Z^*, p) = C_+(m^*, p). \tag{3.3}$$

By the definition of $\mathbf{C}_+$, (3.3) will follow if we fix an arbitrary s such that $\mathbf{P}(Z, s) \geqq p$ and an arbitrary Z in Ξ' and show that

$$\mathbf{C}(Z, s) \geqq C_+(m^*, p). \tag{3.4}$$

As noted before, there is a unique number α such that

$$P(m^*, \alpha) = p \qquad \text{for } 0 < p < 1.$$

Thus, $C_+(m^*, p) = C(m_\alpha^*)$ and (3.4) becomes

$$\mathbf{C}(Z, s) \geqq C(m_\alpha^*). \tag{3.5}$$

Let e and c be defined by (2.3) and (2.4), and let

$$\mathbf{e}(x, q, z) = f(x)qb(z),$$
$$\mathbf{c}(x, q, z) = z/U + T(x)\delta(x)qb(z) \qquad \text{for } x \in R, \quad z \geqq 0, \quad 0 \leqq q \leqq 1.$$

Then (3.5) may be written as

$$\begin{aligned} \int_G \int_R \mathbf{c}(x, Q(x), g(x))\, dx F(dg, s, Z) &\geqq \int_R c(x, m^*(x, \alpha))\, dx \\ &= \int_G \int_R c(x, m^*(x, \alpha))\, dx F(dg, s, Z), \end{aligned} \tag{3.6}$$

where $Q(x) = X(x, s, Z)$. The last equality follows from the fact that the integrand $c(x, m^*(x, \alpha))$ does not depend on g and that $F(\cdot, s, Z)$ is a probability measure. Since we are considering only s such that $\mathbf{P}(Z, s) \geqq p$, we have

$$\int_G \int_R [\mathbf{e}(x, Q(x), g(x)) - e(x, m^*(x, \alpha))]\, dx F(dg, s, Z) \geqq 0.$$

Thus to prove (3.6), it is sufficient to show that there is a $\lambda > 0$ such that for all g in G and x in R,

$$\mathbf{c}(x, Q(x), g(x)) - c(x, m^*(x, \alpha)) \geqq \frac{1}{\lambda}[\mathbf{e}(x, Q(x), g(x)) - e(x, m^*(x, \alpha))]. \tag{3.7}$$

From the previous section we know that for $\lambda = A(U\alpha)$, (2.5) and (2.6) hold with $\lambda_s = \lambda$ and $m^*(x, s) = m^*(x, \alpha)$. These inequalities in turn give, for any g in G,

$$
\begin{aligned}
c(x, g(x)) - c(x, m^*(x, \alpha)) &= \int_{m^*(x,\alpha)}^{g(x)} c'(x, z)\, dz \geqq \frac{1}{\lambda} \int_{m^*(x,\alpha)}^{g(x)} e'(x, z)\, dz \\
&= \frac{1}{\lambda}[e(x, g(x)) - e(x, m^*(x, \alpha))].
\end{aligned}
\tag{3.8}
$$

By taking $g \equiv 0$ in (3.8), we obtain

$$
-c(x, m^*(x, \alpha)) \geqq -\frac{1}{\lambda} e(x, m^*(x, \alpha)). \tag{3.9}
$$

By virtue of (3.8), (3.9), and the fact that $0 \leqq Q(x) \leqq 1$, we have

$$
\begin{aligned}
&\mathbf{c}(x, Q(x), g(x)) - c(x, m^*(x, \alpha)) \\
&\quad = [c(x, g(x)) - c(x, m^*(x, \alpha))]Q(x) - c(x, m^*(x, \alpha))(1 - Q(x)) + \frac{g(x)}{U}(1 - Q(x)) \\
&\quad \geqq \frac{1}{\lambda}\{[e(x, g(x)) - e(x, m^*(x, \alpha))]Q(x) - e(x, m^*(x, \alpha))(1 - Q(x))\} \\
&\quad = \frac{1}{\lambda}[\mathbf{e}(x, Q(x), g(x)) - e(x, m^*(x, \alpha))]
\end{aligned}
$$

for all g in G. Thus, (3.7) holds and consequently (3.3), and we have that immediate investigation coupled with the broad search allocation m^* minimizes the mean time to find the target.

4. Additivity of optimal search plan. In this section we show that if the optimal plan of § 3 is followed and if at any time during the search before the target has been found it is decided to replan the search so as to minimize the mean time *remaining* to find the target, the optimal plan is to continue with the original search plan.

Consider an immediate identification search using the broad search allocation m^*, which has stopped at broad search time r and has not yet found the target. We consider two cases.

For the first case we suppose that no contact has been found at broad search time r. Since contacts are investigated immediately, there are no uninvestigated contacts. Define the functions $\mathbf{c}_r$ and $\mathbf{e}_r$ by

$$
\mathbf{e}_r(x, q, z) = \frac{q}{1 - P(m^*, r)} f(x)[b(m^*(x, r) + z) - b(m^*(x, r))], \tag{4.1}
$$

$$
\mathbf{c}_r(x, q, z) = z/U + qT(x)\delta(x)[b(m^*(x, r) + z) - b(m^*(x, r))]. \tag{4.2}
$$

If $g(x, s)$ is the broad search density function for the search effort added during the broad search interval $[r, r + s]$ and if $Q(x, s)$ is the probability that a contact made at x by broad search time $r + s$ will have been investigated, then

$$\int_R \mathbf{e}_r(x, Q(x, s), g(x, s))\, dx$$

is the probability of having found the target by broad search time $r + s$ and

$$\int_R \mathbf{c}_r(x, Q(x, s), g(x, s))\, dx$$

is the expected amount of time spent investigating false contacts by broad search time $r + s$. Thus, $\mathbf{e}_r$ and $\mathbf{c}_r$ correspond to $\mathbf{e}$ and $\mathbf{c}$ of § 3.

By the same argument as in § 3, we may show that the optimal search plan is immediate contact investigation coupled with the broad search allocation given by

$$m_r^*(x, s) = m^*(x, r + s) - m^*(x, r) \qquad \text{for } s \geqq 0, \quad x \text{ in } R.$$

That is, one continues the original plan.

In the second case, we suppose that a contact has been found at x_0 at broad search time r and has not been investigated. We shall show that any plan which delays investigating this contact can be improved by immediately investigating the contact. This coupled with the above result gives that continuing the original immediate identification plan with broad search density given by m^* is optimal.

In the last section it is shown that the probability that the target is the contact at x_0 is given by

$$\eta = \frac{f(x_0)}{\delta(x_0)(1 - P(m^*, r)) + f(x_0)}. \tag{4.3}$$

The density for the continuous part of the target location distribution is given by

$$f_r(x) = \frac{(1 - \eta) f(x)(1 - b(m^*(x, r)))}{1 - P(m^*, r)} \qquad \text{for } x \in R.$$

Let Z_r be a search plan for continuing the search beyond broad search time r. Let $X_r(x, s, Z_r)$ be the probability, using plan Z_r, that a contact at x will be identified by broad search time $r + s$. For convenience, let $Y(s) = X_r(x_0, s, Z_r)$. The probability that the target has been identified by broad search time $r + s$ is

$$\eta Y(s) + (1 - \eta)\mathbf{P}_r(Z_r, s),$$

where

$$\mathbf{P}_r(Z_r, s) = \int_G \int_R \mathbf{e}_r(x, X_r(x, s, Z_r), g(x))\, dx F(dg, s, Z_r).$$

Let

$$\mathbf{C}_r(Z_r, s) = \int_G \int_R \mathbf{c}_r(x, X_r(x, s, Z_r), g(x))\, dx F(dg, s, Z_r).$$

Then restricting ourselves to Ξ' as before, we find the expected time required for broad search and investigation of false targets to be

$$\mu_r(Z_r) = \int_0^\infty \mathbf{C}_r(Z_r, s)[\eta Y(ds) + (1-\eta)\mathbf{P}_r(Z_r, ds)] + (1-\eta)T(x_0)\int_0^\infty Y(s)\mathbf{P}_r(Z_r, ds).$$

The last term on the right arises from the fact that if the contact at x_0 is a false target and if it is investigated before the target is found, the mean time to investigate it must be counted in $\mu_r(Z_r)$. Modify Z_r by requiring that the contact at x_0 be identified immediately and call the modified plan Z_r^*. We claim that

$$\mu_r(Z_r^*) \leqq \mu_r(Z_r).$$

Observe that

$$\mu_r(Z_r^*) = (1-\eta)\int_0^\infty \mathbf{C}_r(Z_r, s)\mathbf{P}_r(Z_r, ds) + (1-\eta)T(x_0),$$

and that by Fubini's theorem,

$$\begin{aligned}\int_0^\infty Y(s)\mathbf{P}_r(Z_r, ds) &= \int_0^\infty \int_0^s Y(ds')\mathbf{P}_r(Z_r, ds)\\ &= \int_0^\infty \int_{s'}^\infty \mathbf{P}_r(Z_r, ds)Y(ds')\\ &= \int_0^\infty [1 - \mathbf{P}_r(Z_r, s-)]Y(ds)\\ &\geqq 1 - \int_0^\infty \mathbf{P}_r(Z_r, s)Y(ds).\end{aligned}$$

Thus,

(4.4) $$\mu_r(Z_r) - \mu_r(Z_r^*) \geqq \int_0^\infty [\eta\mathbf{C}_r(Z_r, s) - (1-\eta)T(x_0)\mathbf{P}_r(Z_r, s)]Y(ds),$$

and to show $\mu_r(Z_r) \geqq \mu_r(Z_r^*)$, it is sufficient to show that

(4.5) $$\eta\mathbf{C}_r(Z_r, s) - (1-\eta)T(x_0)\mathbf{P}_r(Z_r, s) \geqq 0.$$

By definition of m^*,

$$\begin{aligned}\lambda = A^{-1}(Ur) &= \frac{f(x_0)b'(m^*(x_0, r))}{T(x_0)\delta(x_0)b'(m^*(x_0, r)) + 1/U}\\ &\leqq \frac{f(x_0)}{\delta(x_0)T(x_0)} = \frac{\eta(1 - P(m^*, r))}{(1-\eta)T(x_0)}\end{aligned}$$

with the last equality following from (4.3). Thus by the decreasing nature of b', we have for all $z \geqq m^*(x, r)$,

$$\frac{f(x)b'(z)}{T(x)\delta(x)b'(z) + 1/U} \leqq \lambda,$$

and thus for $0 \leqq Q(x) \leqq 1$ and $z \geqq m^*(x, r)$,

$$\text{(4.6)} \qquad \frac{f(x)b'(z)Q(x)}{T(x)\delta(x)Q(x)b'(z) + 1/U} \leqq \lambda \leqq \frac{\eta(1 - P(m^*, r))}{(1 - \eta)T(x_0)}.$$

From (4.6), we obtain for $z \geqq m^*(x, r)$,

$$\text{(4.7)} \qquad \frac{T(x_0)(1 - \eta)}{1 - P(m^*, r)} f(x)b'(z)X(x, s, Z_r) \leqq \eta\left[T(x)\delta(x)X(x, s, Z_r)b'(z) + \frac{1}{U}\right],$$

and integrating both sides of (4.7) with respect to z over the interval $[m^*(x, r), m^*(x, r) + g(x)]$, we obtain

$$(1 - \eta)T(x_0)\mathbf{e}_r(x, X(x, s, Z_r), g(x)) \leqq \eta\mathbf{c}_r(x, X(x, s, Z_r), g(x)).$$

Integrating both sides of the above inequality over R and G, we find that (4.5) holds. Thus, the contact at x_0 should be investigated immediately, and we have proved the additivity of the optimal search plan.

5. No plan maximizes $P(t)$ for all t. We show by example that in general no plan maximizes $P(t)$, the probability of finding the target by time t, for all $t > 0$, when there are false targets to deal with.

Consider two regions I and II in the plane, each having Lebesgue measure 1. Let

$$f(x) = \begin{cases} 0.1 & \text{for } x \text{ in I}, \\ 0.9 & \text{for } x \text{ in II}, \end{cases} \qquad \delta(x) = \begin{cases} 1 & \text{for } x \text{ in I}, \\ 89 & \text{for } x \text{ in II}. \end{cases}$$

Assume that the time required to identify a contact is exactly one unit of time (i.e., $T = 1$), $U = 1$ and $b(z) = 1 - e^{-z}$. We wish to maximize $P(1 + h)$ for $h > 0$.

Consider two search plans, plan 1 and plan 2, both of which call for immediate investigation of contacts. In plan 1, broad search is confined to region I with density

$$m_1(x, s) = \begin{cases} s & \text{for } x \text{ in I}, \\ 0 & \text{for } x \text{ in II}. \end{cases}$$

In plan 2, broad search is confined to region II with density

$$m_2(x, s) = \begin{cases} 0 & \text{for } x \text{ in I}, \\ s & \text{for } x \text{ in II}. \end{cases}$$

To find the target by time $1 + h$, it must be contacted by time h. Let $P_1(t)$ and $P_2(t)$ be the probability of finding the target by time t in plan 1 and plan 2 respectively.

Then

$$P_1(1+h) \leqq \Pr\{\text{contacting target by broad search time } h\} = 0.1(1 - e^{-h}),$$

and for sufficiently small $h > 0$,

$$\begin{aligned} P_2(1+h) = \Pr\{&\text{contacting target by broad search time } h \text{ and not} \\ &\text{detecting any false targets by broad search time } h\} \\ \geqq\ &0.9(1 - e^{-h})e^{-89h}. \end{aligned}$$

Hence

$$\lim_{h \to 0} P_1(1+h)/P_2(1+h) \leqq 1/9$$

and

$$P_2(1+h) > P_1(1+h) \qquad \text{for all sufficiently small } h.$$

However, one may check that the optimal plan (i.e., the one which produces τ^*, the minimal mean time to find the target) starts searching in area I. Thus the optimal plan coincides with plan 1 over some nondegenerate interval of time $[0, h^*]$. Let $P^*(t)$ be the probability of finding the target by time t using the optimal plan. Then for sufficiently small h, $P^*(1+h) \leqq 0.1(1 - e^{-h}) < P_2(1+h)$. Obviously then, no plan can maximize $P(t)$ for all t, since such a plan would necessarily produce a mean time to find the target which is strictly less than τ^*, which would contradict the minimality of τ^*.

6. Examples. For two search situations, we find m^* in a form which can be computed numerically. Recall that the broad search allocation m^* coupled with immediate identification is the optimal search plan in Ξ.

In the second example, we also show that for a search with circularly symmetric target location distribution having a central tendency and a constant false contact density δ, the rate at which search spreads from the center of the distribution decreases as δ increases.

For the first example, let R be the plane and let the a priori target location density function be given by

$$f(x, y) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{x^2 + y^2}{2\sigma^2}\right) \qquad \text{for } (x, y) \text{ in } R,$$

where $\sigma > 0$. Let T be a constant function and the false contacts density be given by

$$\delta(x, y) = \begin{cases} 0 & \text{for } y \geqq 0, \\ K > 0 & \text{for } y < 0 \end{cases} \quad \text{for } (x, y) \in R.$$

We suppose $b(z) = 1 - e^{-z}$ and find m^* for this situation.

Note that $b'^{-1}(z) = \ln(1/z)$ for $1 \geqq z > 0$. Let

$$h(x, y, \Lambda) = \frac{U}{2\pi\sigma^2\Lambda} \exp\left(-\frac{x^2 + y^2}{2\sigma^2}\right) \qquad \text{for } (x, y) \in R \text{ and } \Lambda > 0.$$

Then

$$d(x, y, \Lambda) = \begin{cases} \ln(h(x, y, \Lambda) - UTK) & \text{if } h(x, y, \Lambda) - UTK > 1 \text{ and } y < 0, \\ \ln(h(x, y, \Lambda)) & \text{if } h(x, y, \Lambda) > 1 \text{ and } y \geqq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Letting $\beta = U/(2\pi\sigma^2)$, one may conclude that

$$A(\Lambda) = \begin{cases} 0 & \text{for } \Lambda \geqq \beta, \\ \dfrac{U}{4\beta}\left[\ln\dfrac{\beta}{\Lambda}\right]^2 & \text{for } \dfrac{\beta}{1 + UTK} \leqq \Lambda \leqq \beta, \\ \dfrac{U}{4\beta}\left[\ln\dfrac{\beta}{\Lambda}\right]^2 + \psi(\Lambda) & \text{for } 0 < \Lambda < \dfrac{\beta}{1 + UTK}, \end{cases}$$

where

$$\psi(\Lambda) = \pi \int_0^{w(\Lambda)} \ln\left[\frac{\beta}{\Lambda}\exp\left(-\frac{r^2}{2\sigma^2}\right) - UTK\right] r\,dr$$

and $w(\Lambda) = \sigma\sqrt{2}[\ln(\beta/\Lambda) - \ln(1 + UTK)]^{1/2}$. Now $m^*(x, y, s) = d(x, y, A^{-1}(Us))$.

In general, we must find A^{-1} numerically; however,

$$A^{-1}(Us) = \beta\exp(-2(\beta s)^{1/2}) \qquad \text{for } 0 < Us \leqq \pi[\sigma\ln(1 + UTK)]^2/2.$$

Thus,

$$m^*(x, y, s) = \begin{cases} 2(\beta s)^{1/2} - (x^2 + y^2)/(2\sigma^2) & \text{for } y \geqq 0 \text{ and } x^2 + y^2 \leqq 4\sigma(Us/2\pi)^{1/2}, \\ 0 & \text{otherwise,} \end{cases}$$

for $0 \leqq s \leqq \pi[\sigma\ln(1 + UTK)]^2/(2U)$.

By examining the form of d given in (6.1), one can see that the broad search begins in the upper half-plane and is allocated to an expanding semicircular region in that half-plane until broad search time $s = \pi\sigma^2[\ln(1 + UTK)]^2/(2U)$. The search then expands into the lower half-plane with broad search effort being placed in both half-planes. The region in which broad search effort is being allocated is then semicircular in both the upper and lower half-planes. However, the radius of the semicircle in the lower half-plane is smaller than its simultaneous counterpart in the upper half-plane. If at any time during this broad search a contact is generated, the broad search stops and the contact is investigated immediately until it is identified as false or the target.

It is interesting to compare the allocation of broad search effort given above to the optimal allocation of effort when the false contact density is 0 everywhere in the plane. In this case, one obtains the Koopman allocation

$$m^*(x, y, s) = \begin{cases} (Us/(\pi\sigma^2))^{1/2} - (x^2 + y^2)/(2\sigma^2) & \text{for } x^2 + y^2 < 2\sigma(Us/\pi)^{1/2}, \\ 0 & \text{otherwise.} \end{cases}$$

Notice that this allocation of effort is circularly symmetric.

For the second example, we consider a search in which R is the plane and in which the a priori target location distribution is circularly symmetric with a central tendency. We further suppose that T and δ are constant functions.

It is clear that search is initiated at the center of the distribution, and that the search spreads out from the center as time passes without finding the target. In addition to finding m^*, we show that as the constant false contact density increases, the rate at which the optimal broad search spreads from the origin decreases.

Let the a priori target location density function f be defined on the plane R so that

$$f(x, y) = g(r),$$

where $r = \sqrt{x^2 + y^2}$. Again let $b(z) = 1 - e^{-z}$. Assume g is strictly decreasing and continuous for $r \geq 0$. To determine m^* we let

$$d(r, \Lambda) = \begin{cases} b'^{-1}\left(\dfrac{\Lambda}{U(g(r) - \Lambda\delta T)}\right) & \text{if } b'(0) \geq \dfrac{\Lambda}{U(g(r) - \Lambda\delta T)} > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$m^*(x, y, s) = d(r, A^{-1}(Us)),$$

where $r = \sqrt{x^2 + y^2}$, and

$$A(\Lambda) = 2\pi \int_0^{\rho(\Lambda)} b'^{-1}\left(\frac{\Lambda}{U(g(r) - \Lambda\delta T)}\right) r\, dr,$$

where

$$\rho(\Lambda) = \begin{cases} g^{-1}[\Lambda(1/(b'(0)U) + \delta T)] & \text{on the domain of } g^{-1}, \\ 0 & \text{otherwise.} \end{cases}$$

Note that g^{-1} is well-defined since g is continuous and strictly decreasing. In general, A and A^{-1} must be computed numerically before m^* can be calculated. Note that $\rho(A^{-1}(Us))$ gives the radius of the disc to which search effort has been applied by broad search time s. Thus, effort is first exerted at a point a distance r_1 from the origin at the broad search time s such that

$$A^{-1}(Us) = g(r_1)/(\delta T + 1/(Ub'(0))).$$

Hence, the density of the effort exerted at a point a distance $r_0 < r_1$ from the origin prior to putting any effort at a distance r_1 from the origin is

$$d\left(r_0, \frac{g(r_1)}{\delta T + 1/(Ub'(0))}\right) = b'^{-1}\left(\frac{b'(0)g(r_1)}{U\delta T b'(0)(g(r_0) - g(r_1)) + g(r_0)}\right)$$

which increases as δ increases. Thus, the rate at which broad search spreads from the origin decreases as δ increases.

7. A posteriori target location distribution. Here we compute the a posteriori target location distribution after search (with false targets) has taken place for a time s. Let us consider the set Ω of searches in which the a priori target location distribution has a probability density function f defined on R. For convenience we take R to be n-dimensional Euclidean space. We suppose that the false contact density function δ is given and that each member of this set of searches has taken place for a fixed broad search time s with the broad search allocation given by m. Moreover, it is assumed that the contacts generated by this search have not been investigated. (In the following we shall suppress the dependence of m, P and φ on s.) Since contacts are not investigated, $P(m)$ becomes the probability of contacting the target.

If no contacts are found, then the a posteriori target distribution has probability density function given by the usual Bayes formula. Here we consider the problem of finding the a posteriori distribution when the broad search has produced exactly N contacts at the points α_i, $i = 1, 2, \cdots, N$. Note that both N and $\alpha = (\alpha_i, \cdots, \alpha_N)$ are random variables. We approach this problem by computing three conditional probabilities which are of interest in themselves and then combine two of them to find the a posteriori target location distribution.

For the convenience of the the reader we shall list the results here before proceeding with the proofs. In contrast to the body of the paper we shall allow the local effectiveness function b to depend on location. That is, the probability of finding the target with z amount of effort given it is located at x is $b(x, z)$. We shall also allow for a different local effectiveness function against false targets by defining $a(x, z)$ to be the probability of contacting a false target located at x with z amount of effort. Let

$$r(x) = \delta(x)a(x, m(x)) \quad \text{for } x \in R,$$

$$u(x) = f(x)b(x, m(x)) \quad \text{for } x \in R,$$

$$q_i(\alpha) = u(\alpha_i) \prod_{j \neq i} r(\alpha_j) \quad \text{for } i = 1, \cdots, N.$$

Then the conditional probability that the ith contact is the target given that one of these contacts represented by α is the target is

$$\gamma_i(\alpha) = \frac{q_i(\alpha)}{\sum_{j=1}^{N} q_j(\alpha)}. \tag{7.1}$$

The conditional probability that the ith contact is the target given α is

$$\eta_i(\alpha) = \frac{q_i(\alpha)}{(1 - P(m)) \prod_{j=1}^{N} r(\alpha_j) + \sum_{j=1}^{N} q_j(\alpha)}. \tag{7.2}$$

The conditional probability that the target is in a set S given α is

$$\frac{\int \int_S f(x)[1 - b(x, m(x))]\, dx \prod_{i=1}^{N} r(\alpha_i)}{(1 - P(m)) \prod_{j=1}^{N} r(\alpha_j) + \sum_{j=1}^{N} q_i(\alpha)} + \sum_{i=1}^{N} \eta_i(\alpha) I(\alpha_i, S), \tag{7.3}$$

where $I(x, S) = 1$ if x is in S and 0 otherwise. When $r(\alpha_i) \neq 0$, for $i = 1, \cdots, N$, (7.1), (7.2) and (7.3) become respectively

$$\gamma_i(\alpha) = \frac{u(\alpha_i)/r(\alpha_i)}{\sum_{j=1}^{N} u(\alpha_j)/r(\alpha_j)},$$

$$\eta_i(\alpha) = \frac{u(\alpha_i)/r(\alpha_i)}{(1 - P(m)) + \sum_{j=1}^{N} u(\alpha_j)/r(\alpha_j)}$$

and

$$\frac{\iint_S f(x)[1 - b(x, m(x))]\, dx}{(1 - P(m)) + \sum_{i=1}^{N} u(\alpha_i)/r(\alpha_i)} + \sum_{i=1}^{N} \eta_i(\alpha) I(\alpha_i, S).$$

We now find the conditional probability that the target is located at α_i given that there are contacts at α_j, $j = 1, \cdots, N$, one of which is the target. Let Ω_k be the subset of searches in Ω which produce $N = k$ contacts, one of which is the target (although one cannot tell which one). We define the defective random variable (i.e., a random variable defined on a space of measure less than 1)

$$\chi_k = (\alpha_1, \cdots, \alpha_k)$$

on Ω_k. The above k-tuple is to be considered unordered in the sense that it has the ordering which would result from "throwing the k contacts in an urn" and designating the ith contact as the one drawn on the ith draw when contacts are drawn out, one at a time, without replacement, in such a fashion that on each draw the remaining contacts all have the same probability of being drawn. The location of the ith contact is given by α_i.

Denote by μ the measure on the Borel sets of R^k which is induced by χ_k. We shall now compute the density function μ' for μ. As was mentioned in the discussion leading up to (1.2), if the number of false targets contacted is $k - 1$, then the distribution of their location is that of $k - 1$ independent draws from a distribution with density

$$\delta(x)a(x, m(x))/\varphi(R) \qquad \text{for } x \in R.$$

Similarly, if the target has been contacted, its location distribution has density

$$f(x)b(x, m(x))/P(m) \qquad \text{for } x \in R.$$

Thus, given that the target is located at α_i and that there are exactly k contacts, the distribution of $(\alpha_1, \cdots, \alpha_k)$ has density

$$\text{(7.4)} \qquad \frac{u(x_i) \prod_{j \neq i} r(x_j)}{P(m)[\varphi(R)]^{k-1}}, \qquad x_i \in R, \quad i = 1, \cdots, k.$$

Since $(\alpha_1, \cdots, \alpha_k)$ is unordered in the sense discussed above,

$$\Pr\{\text{target is } i\text{th contact} | \text{target is one of the } k \text{ contacts}\} = 1/k.$$

Dividing (7.4) by k and summing from $i = 1, \cdots, k$, we obtain

$$\text{(7.5)} \qquad \frac{\sum_{i=1}^{k} u(x_i) \prod_{j \neq i} r(x_j)}{kP(m)[\varphi(R)]^{k-1}}, \qquad x_j \in R, \quad j = 1, \cdots, k,$$

which is the density of $(\alpha_1, \cdots, \alpha_k)$ given that there are k contacts, one of which is the target. Multiplying (7.5) by the probability of k contacts, one of which is the target, we obtain

$$\mu'(x_1, \cdots, x_k) = \frac{e^{-\varphi(R)}}{k!} \sum_{i=1}^{k} u(x_i) \prod_{j \neq i} r(x_j) \tag{7.6}$$

for $x_i \in R$, $i = 1, \cdots, k$, the density of μ.

Let $B = B_1 \times \cdots \times B_k$ be a Borel subset of R^k. Then using (7.1) and (7.6), one may compute

$$\int_B \gamma_i(x_1, \cdots, x_k)\mu'(x_1, \cdots, x_k)\, dx_1, \cdots, dx_k$$

$$= \frac{e^{-\varphi(R)}}{(k-1)!} \frac{1}{k} \int_{B_i} u(x_i)\, dx_i \prod_{j \neq i} \int_{B_j} r(x_j)\, dx_j$$

$$= \Pr \{\text{there are exactly } k \text{ contacts } (\alpha_1, \cdots, \alpha_k) \in B \text{ and the target is contacted at } \alpha_i\}.$$

Thus it follows from the definition of conditional probability given in [5, Chap. I, § 7] that $\gamma_i(\alpha)$ is the conditional probability that the target is located at α_i given $\chi_k = (\alpha_1, \cdots, \alpha_k)$ (i.e., given there are k contacts located at α_i, $i = 1, \cdots, k$, one of which is the target).

We now consider the problem of computing the probability that the target is located at α_i given there are k contacts at $\alpha_1, \cdots, \alpha_k$. Note that we do not condition this on one of the contacts being the target.

Letting Λ_k be the restriction of α to the set of searches which produce exactly k contacts, we find, in a manner similar to the above, that the distribution of Λ_k has density

$$\mu^*(x_1, \cdots, x_k) = \frac{e^{-\varphi(R)}}{k!} \left[\sum_{i=1}^{k} u(x_i) \prod_{j \neq i} r(x_j) + [1 - P(m)] \prod_{i=1}^{k} r(x_i) \right].$$

Then using (7.2), one may calculate that for any Borel subset $B = B_1 \times \cdots \times B_k$ of R^k,

$$\int_B \eta_i(x_1, \cdots, x_k)\, d\mu^*(x_1, \cdots, x_k)$$

$$= \frac{e^{-\varphi(R)}}{(k-1)!} \frac{1}{k} \int_{B_i} u(x_i)\, dx_i \prod_{j \neq i} \int_{B_j} r(x_j)\, dx_j$$

$$= \Pr \{\text{there are exactly } k \text{ contacts } (\alpha_1, \cdots, \alpha_k) \in B \text{ and the target is contacted at } \alpha_i\}.$$

This proves that $\eta_i(\alpha)$ is the conditional probability of the target being located at α_i given $\Lambda_k = (\alpha_1, \cdots, \alpha_k)$ (i.e., given there are exactly k contacts located at $(\alpha_1, \cdots, \alpha_k)$).

That the posterior target location distribution is given by (7.3) follows readily upon observing that by an argument similar to the ones given above,

$$\frac{f(x)[1 - b(x, m(x))] \prod_{i=1}^{k} r(\alpha_i)}{[1 - P(m)] \prod_{i=1}^{k} r(\alpha_i) + \sum_{i=1}^{k} q_i(\alpha)}$$

is the density for the conditional joint probability that the target is located in a subset of R and is not one of the contacts given by $\Lambda_k = (\alpha_1, \cdots, \alpha_k)$.

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