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OPTIMAL SEARCH IN THE PRESENCE OF POISSON-DISTRIBUTED FALSE TARGETS*

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Abstract. This paper finds search plans, which are optimal in the sense of minimizing mean time to find the target, for several types of searches for a stationary target, in the presence of Poisson-distributed false targets. One must allocate the available effort between broad search to develop contacts and contact investigation to ascertain whether these contacts are real or false targets; one must further allocate within each of these two types of effort. A search plan is described by a pair of functions (m, λ), where m gives the broad search effort density as a function of broad search time and location, and λ gives the amount of contact investigation effort that one is willing to exert in attempting to identify a contact as a function of the location of the contact and broad search time. Two general optimization theorems presented in this paper are applied to find optimal search plans. Both theorems may be thought of as extensions of a Neyman–Pearson technique to allocations of more than one type of resource.

1. Introduction. In this paper we continue the work started in [5] where the optimal search plan is found for a class of searches for a stationary target, when the search is complicated by the occurrence of false targets. The search plans considered there are subject to the restriction that once a contact investigation is begun, it must not be interrupted until the contact is identified. In this paper we make a major departure from the assumptions of [5]; in particular, we do not require uninterrupted contact investigation. As in [5], we find the plan which minimizes the mean time to find the target among the class of plans Φ defined below. We note, however, that the class Φ considered here is more restrictive than its counterpart in [5].

We consider search for a stationary target whose a priori location distribution has a density function f defined on a region R of Euclidean m-space. The search consists of exerting two types of effort; namely, broad search and contact investigation effort. The *broad search* effort is applied by a sensor which is capable of detecting but not identifying the target. The search is complicated by false targets. That is, the broad search sensor may detect objects which cannot be distinguished from the target without further investigation. Any such object which is detected is called a *contact* and any such object which is not the target is called a *false target.*

In order to identify a contact as the target or a false target, a contact investigation is required. Effort directed toward identifying a contact is called *contact investigation* effort. We assume that investigation of one contact makes no contribution to the investigation of any other contact or to the broad search.

We distinguish two types of search time. Cumulative broad search time will be denoted by s. Cumulative time spent in all aspects of search and investigation will be denoted by t. For clarity, we say that the target has been *contacted* when it

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appears as a contact and that it has been *identified* when the contact which is the target is identified by contact investigation. We say the target has been *found* when it has been contacted and identified.

Throughout this paper we shall speak of "search effort" without defining the term. This is to allow the model to fit many search situations. Possible definitions of effort are track length, man hours, etc. When we speak of an *effort density* function defined on R, we mean a function h such that for any Borel set $R' \subset R$,

$$\int_{R'} h(x)\,dx = \text{amount of effort placed in } R'.$$

For each $x \in R$, we define $U(x) > 0$ to be the *broad search rate* at x. That is, the time required to apply effort according to the effort density function h is

$$\int_R \frac{h(x)}{U(x)}\,dx.$$

The class Φ of search plans considered here is now described. A member of Φ specifies (1) broad search effort density as a function of broad search time and location only and (2) the amount of contact investigation which one is willing to exert at a point by broad search time s. When we say that the searcher is *willing* to exert z amount of contact investigation effort at a point x, we mean that in the event a contact is made at x, the searcher will investigate that contact until he either identifies the contact or exerts z amount of effort. As indicated, this will be a function of the amount of broad search time that has transpired. Usually as more broad search is performed, one is willing to exert more contact investigation effort. If no contact is made at point x, then no contact investigation effort is applied there.

Mathematically, a member of Φ is a pair of functions (m, λ), where m and λ are defined on $R \times [0, \infty)$ such that $m(x, s)$ gives the broad search effort density at x by broad search time s, and $\lambda(x, s)$ gives the amount of contact investigation effort one is willing to exert at x by broad search time s. In addition, we require that

$$\int_R \frac{m(x, s)}{U(x)}\,dx = s \quad \text{for } s \geqq 0$$

and that $m(x, 0) = 0$, $m(x, \cdot)$ and $\lambda(x, \cdot)$ be monotone increasing, and $\lambda(x, \cdot)$ be continuous for $x \in R$.

The condition that $\lambda(x, \cdot)$ be continuous is imposed to assure that $I(m, \lambda, s)$, defined later in this section, correctly gives the expected amount of time spent investigating false targets by broad search time s, given that the target has not been found before broad search time s. If $\lambda(x, \cdot)$ were allowed to have jumps, it would be necessary to specify the order in which contacts are investigated at broad search times at which more than one contact was present and a jump in contact investigation effort was prescribed for both contacts. This would introduce difficulties which are not easily handled by the methods of this paper.

We remark that the other restrictions imposed upon Φ are not the most natural in the sense that m and λ are allowed to depend only on broad search time and location and not, for example, on the number of contacts. This restriction on the class Φ is necessary to retain the separable nature of the optimal allocation problem so that the optimal plan (in the sense of minimizing mean time to find the target) in Φ can be found by the methods developed in § 3.

For the broad search, we assume that there is a *target local effectiveness* function b defined on $R \times [0, \infty)$ such that for each x in R and $z \geqq 0$, $b(x, z)$ is the conditional probability of contacting the target given that the target is located at x and the broad search effort density is z at x. Furthermore, $b(x, \cdot)$ is a probability distribution function. Thus, the probability of contacting the target by broad search time s using the broad search effort density m is given by

$$\int_R f(x)b(x, m(x, s))\, dx. \tag{1.1}$$

For false targets we assume there is a *false target local effectiveness function* $\tilde{b}$ defined on $R \times [0, \infty)$ such that for each x in R and $z \geqq 0$, $\tilde{b}(x, z)$ is the conditional probability of contacting a false target which is located at x given that the broad search effort density is z at x. Moreover, there is a nonnegative Borel measurable function δ defined on R called the *false target density*. This function is defined so that for any Borel measurable region $R' \subset R$, the expected number of false targets contacted in R' by broad search time s is given by

$$\varphi(R', s) \equiv \int_{R'} \delta(x)\tilde{b}(x, m(x, s))\, dx < \infty.$$

Let $X(R', s)$ be the number of false targets detected in R' by broad search time s. We assume that the contacting of false targets is independent of contacting the target and that for Borel subsets $R' \subset R$,

$$\Pr\{X(R', s) = n\} = \frac{[\varphi(R', s)]^n e^{-\varphi(R', s)}}{n!} \quad \text{for } n = 0, 1, \cdots. \tag{1.2}$$

In addition we assume that for any Borel set $R' \subset R$, $X(R', \cdot)$ is a nonhomogeneous Poisson process in time, i.e., for $s \geqq r \geqq 0$,

$$\Pr\{X(R', r) - X(R', s) = n\} = \exp[-\psi(r, s)][\psi(r, s)]^n/n! \quad \text{for } n = 0, 1, 2, \cdots,$$

where $\psi(r, s) = \varphi(R', r) - \varphi(R', s)$. Similarly, for any broad search time s, $X(\cdot, s)$ is assumed to be a nonhomogeneous Poisson process in space. See [1, p. 337] for a discussion of these processes in the homogeneous case. It follows from this assumption that if R_1 and R_2 are disjoint Borel subsets of R, then $X(R_1, s)$ and $X(R_2, s)$ are independent and $X(R_1 \cup R_2, s) = X(R_1, s) + X(R_2, s)$.

Our assumptions also imply a spatial distribution for the false targets as follows. Set $\tilde{b}(x, m(x, s)) = 1$, in the integral defining φ. Then all false targets are detected, and (1.2) gives the probability of there being n false targets in the region R'. Let

$$D(R') = \int_{R'} \delta(x)\, dx.$$

Then by an argument similar to that given in [1, p. 343], the conditional distribution of the false target locations, given there are exactly n false targets in R', is that of n independent draws from a probability distribution which has density

$$\frac{\delta(x)}{D(R')} \quad \text{for } x \in R'.$$

We assume there is a function a such that $a(x, w)$ is the probability of identifying a contact at x which is the target, with w amount of contact investigation effort. We also assume that $a(x, \cdot)$ is a probability distribution function. For $x \in R$, let $\Lambda(x) > 0$ be the *contact investigation rate* at x. That is, it requires $w/\Lambda(x)$ amount of time to apply w amount of contact investigation effort at x.

The conditional probability of having identified the target by broad search time s given (i) the target is at x, (ii) the target has been contacted by broad search time s, and (iii) one is willing to exert $\lambda(x, s)$ investigation effort, is $a(x, \lambda(x, s))$. Thus the probability of having identified the target by broad search time s using the search plan (m, λ) is

$$P(m, \lambda, s) \equiv \int_R f(x)b(x, m(x, s))a(x, \lambda(x, s))\, dx. \tag{1.3}$$

For false targets, we assume that there is a function A such that $A(x, w)$ is the probability of identifying a false target contacted at $x \in R$ with w amount of contact investigation effort. The contact investigation rate $\Lambda(x)$ remains the same as with the contact which is the target. The expected amount of time spent investigating a false target at x when one is willing to exert $\lambda(x, s)$ investigation effort by broad search time s, given that a contact has been made at x, is $\alpha(x, \lambda(x, s))/\Lambda(x)$, where

$$\alpha(x, w) \equiv \int_0^w [1 - A(x, u)]\, du \quad \text{for } w \geqq 0.$$

Henceforth, whenever it is convenient, we shall suppress the dependence of δ, U, Λ, b, $\tilde{b}$, a, A, and α on x.

It has been demonstrated in [5] that in general one cannot find a plan which maximizes the probability of finding the target at each instant of time during the search. We thus define an optimal plan as one that minimizes the mean time to find the target.

The expected time required to contact and identify the target is $E[\tau_s] + E[\tau_c] + E[T]$, where τ_s is the amount of time spent in broad search before the target is identified, τ_c is the amount of time spent investigating false targets, and T is the amount of time required to investigate the contact which is the target. The optimal search plan (m^*, λ^*) in Φ is the plan which minimizes the expected time to find the target. However, since $E[T]$ is the same for all plans, minimizing the expected time to find the target is equivalent to minimizing μ, where

$$\mu(m, \lambda) = E[\tau_s] + E[\tau_c]. \tag{1.4}$$

Define for (m, λ) in Φ and $s \geqq 0$,

$$I(m, \lambda, s) \equiv \int_R \delta(x)\tilde{b}(x, m(x, s))\frac{\alpha(x, \lambda(x, s))}{\Lambda(x)}\, dx.$$

Since the occurrence of false targets is governed by (1.2), we have by [5, p. 244] that $I(m, \lambda, s)$ is the expected amount of time spent investigating false targets up to broad search time s (given that the target has not been found) using the plan (m, λ). Thus

$$E[\tau_c] = \int_0^\infty I(m, \lambda, s)P(m, \lambda, ds) \quad \text{for } (m, \lambda) \text{ in } \Phi.$$

Let $C(m, \lambda, s) \equiv s + I(m, \lambda, s)$. Then $C(m, \lambda, s)$ is the total expected time spent in broad search and contact investigation by broad search time s (given that the target has not been found) using the plan (m, λ) in Φ. Since

$$E[\tau_s] = \int_0^\infty sP(m, \lambda, ds), \tag{1.5}$$

we have

$$\mu(m, \lambda) = \int_0^\infty C(m, \lambda, s)P(m, \lambda, ds) \quad \text{for } (m, \lambda) \text{ in } \Phi. \tag{1.6}$$

Thus, our object is to find $(m^*, \lambda^*) \in \Phi$ such that $\mu(m^*, \lambda^*) \leqq \mu(m, \lambda)$ for $(m, \lambda) \in \Phi$. We do this by solving the following intermediate problem: Let G be the set of pairs of nonnegative Borel measurable functions from R into $[0, \infty)$. Define

$$p(x, z, w) = f(x)b(z)a(w), \tag{1.7}$$

$$c(x, z, w) = \frac{z}{U} + \frac{\delta}{\Lambda}\tilde{b}(z)\alpha(w) \quad \text{for } x \in R, \quad z \geqq 0, \quad w \geqq 0, \tag{1.8}$$

and for $(d, g) \in G$,

$$\mathbf{P}(d, g) \equiv \int_R p(x, d(x), g(x))\, dx, \tag{1.9}$$

$$\mathbf{C}(d, g) \equiv \int_R c(x, d(x), g(x))\, dx. \tag{1.10}$$

Then for each $s \geqq 0$, we seek $m^*(\cdot, s)$ and $\lambda^*(\cdot, s)$ such that $\int_R (m^*(x, s)/U)\, dx = s$ and for any $(d, g) \in G$,

$$\mathbf{P}(d, g) \geqq \mathbf{P}(m^*(\cdot, s), \lambda^*(\cdot, s)) \quad \text{implies} \quad \mathbf{C}(d, g) \geqq \mathbf{C}(m^*(\cdot, s), \lambda^*(\cdot, s)). \tag{1.11}$$

That is, $(m^*(\cdot, s), \lambda^*(\cdot, s))$ gives the greatest probability of detection for the expected cost $\mathbf{C}(m^*(\cdot, s), \lambda^*(\cdot, s)) = C(m^*, \lambda^*, s)$. We shall find $m^*(\cdot, s)$ and $\lambda^*(\cdot, s)$ with the aid of the optimization theorems given in § 3. Having done this so that $(m^*, \lambda^*) \in \Phi$, it follows in the same manner as in [5, p. 248] that $\mu(m^*, \lambda^*) \leqq \mu(m, \lambda)$ for $(m, \lambda) \in \Phi$.

For convenience, we state the above as follows.

PROPOSITION 1.1. *If* $(m^*, \lambda^*) \in \Phi$ *such that* $P(m^*, \lambda^*, \cdot)$ *is continuous and strictly increasing and satisfies* (1.11) *for all* $s > 0$, *then*

$$\mu(m^*, \lambda^*) \leqq \mu(m, \lambda) \quad \textit{for } (m, \lambda) \in \Phi.$$

2. Summary. Section 3 contains two theorems on allocating more than one type of resource to optimize a separable nonlinear functional subject to a constraint. Theorem 3.1 deals with allocating two types of resources and is specifically aimed at solving search problems in which one must continuously split his effort between contact investigation and broad search.

Theorem 3.2 gives a method for solving optimization problems involving the choice of an n-tuple of extended real-valued functions by reducing the problem to a sequence of two optimization problems in which one first chooses an n-tuple of real numbers to minimize cost for a given effectiveness and then solves a functional constrained maximization problem using Neyman–Pearson techniques.

When $n = 1$, Theorem 3.2 becomes essentially the nonlinear functional form of the Neyman–Pearson lemma given in [6]. In the present formulation, Theorem 3.2 has the advantage of combining the discrete and continuous allocation theorems of [6] into one theorem. The remarks (1), (2), (3), (4) and (6) in [6] and the discussion of the relationship to Lagrange multipliers on page 58 of [6] are applicable, with the obvious modifications, to Theorem 3.2.

In § 4, Theorems 4.1 and 4.2, which are extensions of Theorem 2 of [5], give optimal plans for two classes of search which satisfy the conditions of § 1. In the first class, the optimal plan calls for continuously dividing effort between contact investigation and broad search. For the second class, the optimal plan calls for immediate and uninterrupted contact investigation until contacts are identified.

In particular, Theorem 4.1 finds the optimal plan (m^*, λ^*) under the following conditions:

(a) $\tilde{b} = b$ and b' exists, is positive, continuous, and strictly decreasing;

(b) A is continuous so that $\alpha'(w) = 1 - A(w)$ for $w \geqq 0$;

(c) a' exists and $\zeta \equiv a'/(1 - A)$ is continuous, positive, and strictly decreasing;

(2.1)

(d) at least one of the following holds:

(i) the mean time to identify any false target is finite, i.e., $\lim_{w \to \infty} \alpha(w) < \infty$;

(ii) there exist $J > 0$ and $\varepsilon > 0$ such that $\zeta(w) \leqq J/w^{1+\varepsilon}$.

Condition (a) states that the local effectiveness function is the same against both false targets and the real target (i.e., $\tilde{b} = b$). It further specifies that this effectiveness function satisfies a "law of diminishing returns." Condition (b) is a technical one which allows us to perform the differentiations required to apply Theorem 3.1. A "law of diminishing returns" for contact investigation is specified by condition (c). In this regard, the function ζ may be thought of as a generalized success rate for contact investigation. This is most clearly seen when $A = a$, in which case $h\zeta(w) + o(h)$ gives the probability of identifying a contact with an increment h of contact investigation effort given w amount of effort has been expended without success. It is the combination of conditions (a) and (c) which forces effort to be allocated simultaneously to contact investigation and broad search. Condition (d) is used to guarantee that one has a balance between broad search and contact investigation effort. That is, one is not required to exert an infinite amount of contact investigation effort to match a finite amount of broad search effort.

Theorem 4.2 finds the optimal plan (m^*, λ^*) when conditions (2.2) below hold. Moreover, it shows that $\lambda^*(x, s) = \infty$ for $x \in R$ and $s > 0$. That is, one investigates contacts immediately, and once the investigation has begun, it continues until the contact is identified. Conditions (2.2) are:

(a) both b and $\tilde{b}$ have derivatives which are positive, continuous, and strictly decreasing;

(2.2) (b) $\tilde{b}/b$ and $\tilde{b}'/b'$ are nondecreasing on $(0, \infty)$; and

(c) α/a is nonincreasing on $(0, \infty)$.

Note that (c) of (2.2) and (c) of (2.1) are mutually exclusive, which indicates that under (2.2), the "law of diminishing returns" does not hold for contact investigation. This motivates the conclusion proved in Theorem 4.2 that one should investigate contacts until identified before returning to broad search.

In § 5 we consider the problem of optimal search when sensor effectiveness is related to sweep width (see [2] for a definition and discussion of sweep width). The sweep width is stochastic with known prior distribution but does not vary in time. It is shown that this problem may be reduced to the search problem described in § 1. By using Theorem 4.2, the optimal plan (m^*, λ^*) is found for a search example involving false targets and a stochastic sweep width. After specializing the example to the case when there are no false targets, the mean time to find the target, when using the optimal plan, is calculated explicitly (see (5.4)).

3. Optimization theorems. In this section we present two optimization theorems which will be used to find optimal search plans in the following sections. The reader exclusively interested in search may wish to proceed directly to the next section and refer to the results of this section when needed.

We indicate the partial derivative of a function g with respect to its ith variable by $D_i g$. As before, R is a region in Euclidean m-space.

THEOREM 3.1. *Let $e(x, z, w)$ and $c(x, z, w)$ be defined as extended real numbers for $x \in R$ and $z, w \in [\mathbf{L}(x), \mathbf{U}(x)]$, where $\mathbf{L}(x)$ and $\mathbf{U}(x)$ are extended real numbers such that $\mathbf{L}(x) < \mathbf{U}(x)$. Assume e and c are Borel measurable. Let G be the set of pairs (d, g) of Borel measurable functions defined on R such that*

$$\mathbf{L}(x) \leqq d(x) \leqq \mathbf{U}(x) \quad \text{and} \quad \mathbf{L}(x) \leqq g(x) \leqq \mathbf{U}(x) \quad \text{for all } x \in R$$

and

$$E(d, g) \equiv \int_R e(x, d(x), g(x))\, dx,$$

$$C(d, g) \equiv \int_R c(x, d(x), g(x))\, dx$$

exist and are finite.

Suppose

(a) $D_i e$ *and* $D_i c$ *exist and are positive for* $i = 2, 3$;

(b) *for each* $x \in R$ *and* $\mathbf{L}(x) \leqq w \leqq \mathbf{U}(x)$, $D_2 e(x, \cdot, w)$ *and* $D_2 c(x, \cdot, w)$ *are Riemann integrable on bounded subintervals of* $[\mathbf{L}(x), \mathbf{U}(x)]$;

(c) *for each* $x \in R$ *and* $\mathbf{L}(x) \leqq z \leqq \mathbf{U}(x)$, $D_3 e(x, z, \cdot)$ *and* $D_3 c(x, z, \cdot)$ *are Riemann integrable on bounded subintervals of* $[\mathbf{L}(x), \mathbf{U}(x)]$;

(d) *there exists a function* β *such that*

$$D_3 e(x, z, w) = \beta(x, w) D_3 c(x, z, w)$$

for $x \in R$ *and* z, w *in* $[\mathbf{L}(x), \mathbf{U}(x)]$.

Suppose that there exist a number $k > 0$ *and* $(d^*, g^*) \in G$ *such that for each* $x \in R$,

(i) $D_2 e(x, z, g^*(x)) \leqq k D_2 c(x, z, g^*(x))$ *for* $d^*(x) < z < \mathbf{U}(x)$,

(ii) $D_2 e(x, z, g^*(x)) \geqq k D_2 c(x, z, g^*(x))$ *for* $\mathbf{L}(x) < z < d^*(x)$,

(iii) $D_3 e(x, d^*(x), w) \leqq k D_3 c(x, d^*(x), w)$ *for* $g^*(x) < w < \mathbf{U}(x)$,

(iv) $D_3 e(x, d^*(x), w) \geqq k D_3 c(x, d^*(x), w)$ *for* $\mathbf{L}(x) < w < g^*(x)$.

Then for $(d, g) \in G$,

$$E(d, g) \geqq E(d^*, g^*) \quad \text{implies} \quad C(d, g) \geqq C(d^*, g^*).$$

Proof. It suffices to show that for each $x \in R$,

$$c(x, d(x), g(x)) - c(x, d^*(x), g^*(x)) \geqq \frac{1}{k}[e(x, d(x), g(x)) - e(x, d^*(x), g^*(x))]. \tag{3.1}$$

To do this, we note that

$$c(x, d(x), g(x)) - c(x, d(x), g^*(x)) = \int_{g^*(x)}^{g(x)} D_3 c(x, d(x), w)\, dw \tag{3.2}$$

and

$$c(x, d(x), g^*(x)) - c(x, d^*(x), g^*(x)) = \int_{d^*(x)}^{d(x)} D_2 c(x, z, g^*(x))\, dz. \tag{3.3}$$

Adding (3.2) and (3.3), we have

$$\begin{aligned} c(x, d(x), g(x)) - c(x, d^*(x), g^*(x)) &= \int_{g^*(x)}^{g(x)} D_3 c(x, d(x), w)\, dw \\ &\quad + \int_{d^*(x)}^{d(x)} D_2 c(x, z, g^*(x))\, dz. \end{aligned} \tag{3.4}$$

By condition (i) and (ii) according as $d(x) \geqq d^*(x)$ or $d(x) < d^*(x)$, we have

$$\begin{aligned} \int_{d^*(x)}^{d(x)} D_2 c(x, z, g^*(x))\, dz &\geqq \frac{1}{k} \int_{d^*(x)}^{d(x)} D_2 e(x, z, g^*(x))\, dz \\ &\geqq \frac{1}{k}[e(x, d(x), g^*(x)) - e(x, d^*(x), g^*(x))]. \end{aligned} \tag{3.5}$$

From (iii) or (iv) and noting that $D_3 e / D_3 c$ is independent of the second variable, we have

$$\int_{g^*(x)}^{g(x)} D_3 c(x, d(x), w)\, dw \geqq \frac{1}{k}[e(x, d(x), g(x)) - e(x, d(x), g^*(x))]. \tag{3.6}$$

Then (3.1) follows by substitution of the sum of (3.5) and (3.6) into (3.4). This proves the theorem.

Next we prove an optimal allocation theorem which is applicable to both continuous and discrete allocation problems where more than one type of resource is to be allocated.

For each $x \in R$, let $J(x)$ be a subset of Euclidean n-space and let $e(x, \cdot, \cdots, \cdot)$ and $c(x, \cdot, \cdots, \cdot)$ be real-valued functions defined on $J(x)$. Let G be the set of functions $g = (g_1, \cdots, g_n)$ defined on R such that $(g_1(x), \cdots, g_n(x)) \in J(x)$ and

$$E(g) \equiv \int_R e(x, g_1(x), \cdots, g_n(x))\, dx,$$

$$C(g) \equiv \int_P c(x, g_1(x), \cdots, g_n(x))\, dx$$

exist and are finite. Define

$$\Delta(x) = \{q | q = e(x, z_1, \cdots, z_n) \text{ for some } (z_1, \cdots, z_n) \text{ in } J(x)\}.$$

THEOREM 3.2. *Suppose that for each* $x \in R$ *and* $q \in \Delta(x)$, *we may define* $(\varphi_1(x, q), \cdots, \varphi_n(x, q))$ *such that*

(a) $(\varphi_1(x, q), \cdots, \varphi_n(x, q)) \in J(x)$,

(b) $e(x, \varphi_1(x, q), \cdots, \varphi_n(x, q)) = q$,

(c) $c(x, \varphi_1(x, q), \cdots, \varphi_n(x, q))$
$= \min \{c(x, z_1, \cdots, z_n) | (z_1, \cdots, z_n) \in J(x) \text{ and } e(x, z_1, \cdots, z_n) = q\}$
$\equiv \xi(x, q)$.

Suppose, in addition, that there exist a $k > 0$ *and a function* Q *defined on* R *such that for* r *and* q *in* $\Delta(x)$, *the following hold*:

$$q - r \geqq k(\xi(x, q) - \xi(x, r)) \quad \text{for } r \leqq q \leqq Q(x), \tag{3.7}$$

$$q - r \leqq k(\xi(x, q) - \xi(x, r)) \quad \text{for } Q(x) \leqq r \leqq q. \tag{3.8}$$

Then $g^* \equiv (g_1^*, \cdots, g_n^*)$, *where*

$$g_i^*(x) \equiv \varphi_i(x, Q(x)), \qquad i = 1, 2, \cdots, n,$$

has the property that for any $g \in G$,

$$E(g) \geqq E(g^*) \quad \text{implies} \quad C(g) \geqq C(g^*).$$

Proof. Fix x in R and let

$$q_2 = e(x, g_1(x), \cdots, g_n(x)) \quad \text{and} \quad q_1 = Q(x) = e(x, g_1^*(x), \cdots, g_n^*(x)).$$

Then

$$\begin{aligned} &c(x, g_1(x), \cdots, g_n(x)) - c(x, g_1^*(x), \cdots, g_n^*(x)) \\ &\quad \geqq c(x, \varphi_1(x, q_2), \cdots, \varphi_n(x, q_2)) - c(x, \varphi_1(x, q_1), \cdots, \varphi_n(x, q_1)) \\ &\quad = \xi(x, q_2) - \xi(x, q_1). \end{aligned}$$

Applying (3.7) or (3.8) according as $q_2 \geqq q_1$ or $q_2 < q_1$, we obtain

$$\xi(x, q_2) - \xi(x, q_1) \geqq \frac{1}{k}(q_2 - q_1)$$
$$\geqq \frac{1}{k}[e(x, g_1(x), \cdots, g_n(x)) - e(x, g_1^*(x), \cdots, g_n^*(x))].$$

Thus, $C(g) - C(g^*) \geqq [E(g) - E(g^*)]/k$, and the theorem is proved.

We remark that in the case where, for a given $x \in R$, $\Delta(x)$ is an interval with lower and upper endpoints $\mathbf{l}(x)$ and $\mathbf{u}(x)$ respectively and $\xi'(x, \cdot)$, the derivative of $\xi(x, \cdot)$, exists and is Riemann integrable, the conditions

$$1 \geqq k\xi'(x, q) \quad \text{for } \mathbf{l}(x) < q < Q(x),$$
$$1 \leqq k\xi'(x, q) \quad \text{for } Q(x) < q < \mathbf{u}(x)$$

are sufficient to guarantee that (3.7) and (3.8) hold.

Similarly, if, for a given $x \in R$, $\Delta(x)$ is countable and can be indexed so that for some interval $[\mathbf{l}(x), \mathbf{u}(x)]$ of extended real numbers, $\Delta(x) = \{q_j | j$ is an integer and $\mathbf{l}(x) < j < \mathbf{u}(x)\}$ and $q_{j-1} < q_j$ for $\mathbf{l}(x) < j - 1 < j < \mathbf{u}(x)$, then the conditions

$$q_j - q_{j-1} \geqq k[\xi(x, q_j) - \xi(x, q_{j-1})] \quad \text{for } \mathbf{l}(x) < j \leqq Q(x),$$
$$q_j - q_{j-1} \leqq k[\xi(x, q_j) - \xi(x, q_{j-1})] \quad \text{for } Q(x) < j \leqq \mathbf{u}(x)$$

are sufficient to guarantee that (3.7) and (3.8) hold.

Corollary 3.2 given next is particularly convenient for the application in § 4 to find the optimal allocation of search effort when immediate and uninterrupted contact investigation is part of the optimal plan. This corollary may easily be extended to allocating n resources, where $n \geqq 2$.

COROLLARY 3.2. *Let e, c, E, C, $\mathbf{L}$, $\mathbf{U}$, and G be defined as in Theorem* 3.1 *and define $J(x)$ to be $[\mathbf{L}(x), \mathbf{U}(x)] \times [\mathbf{L}(x), \mathbf{U}(x)]$ for $x \in R$. Let*

$$\Delta(x) = \{q | q = e(x, z, w) \text{ for some } (z, w) \in J(x)\} \quad \text{for } x \in R.$$

Suppose that for each x in R, $\Delta(x)$ is an interval with lower and upper endpoints $\mathbf{l}(x)$ and $\mathbf{u}(x)$ respectively and that there exist an extended real number $g^*(x)$ and a differentiable function $\varphi_1(x, \cdot)$ defined on $\Delta(x)$ such that

$$e(x, \varphi_1(x, q), g^*(x)) = q$$

and

$$\xi(x, q) \equiv c(x, \varphi_1(x, q), g^*(x)) = \min \{c(x, z, w) | e(x, z, w) = q \text{ and } (z, w) \in J(x)\}.$$

Suppose, furthermore, that $\varphi_1(x, \cdot)$ is a strictly increasing function and that $\xi'(x, \cdot)$, the derivative of $\xi(x, \cdot)$, exists and is Riemann integrable.

Define $\hat{c}(x, z) = c(x, z, g^*(x))$ and $\hat{e}(x, z) = e(x, z, g^*(x))$. If $\hat{e}(x, \cdot)$ is strictly increasing and if there exist a $k > 0$ and a function d^* defined on R such that $(d^*, g^*) \in G$ and

$$D_2\hat{e}(x, z) \geqq kD_2\hat{c}(x, z) \quad \text{for } \mathbf{L}(x) < z < d^*(x), \tag{3.9}$$

$$D_2\hat{e}(x, z) \leqq kD_2\hat{c}(x, z) \quad \text{for } d^*(x) < z < \mathbf{U}(x), \quad \text{for } x \in R, \tag{3.10}$$

then (d^*, g^*) *has the property that for any* $(d, g) \in G$,

$$E(d, g) \geqq E(d^*, g^*) \quad \textit{implies} \quad C(d, g) \geqq C(d^*, g^*).$$

Proof. Conditions (a) through (c) of Theorem 3.2 clearly hold with $n = 2$ and $\varphi_2(x, \cdot)$ replaced by $g^*(x)$. Let $Q(x) = \hat{e}(x, d^*(x))$. Since $\hat{e}(x, \cdot)$ is strictly increasing, $\varphi_1(x, Q(x)) = d^*(x)$. Since $\varphi_1(x, \cdot)$ is strictly increasing, (3.9) and (3.10) become

(3.11) $$D_2\hat{e}(x, \varphi_1(x, q)) \geqq kD_2\hat{c}(x, \varphi_1(x, q)) \quad \text{for } \mathbf{l}(x) \leqq q < Q(x),$$

(3.12) $$D_2\hat{e}(x, \varphi_1(x, q)) \leqq kD_2\hat{c}(x, \varphi_1(x, q)) \quad \text{for } Q(x) < q \leqq \mathbf{u}(x).$$

Since $\xi'(x, q) = D_2\hat{c}(x, \varphi_1(x, q))\varphi_1'(x, q)$ and $\hat{e}(x, \varphi_1(x, q)) = q$, (3.7) and (3.8) hold, and the corollary follows from Theorem 3.2.

Since the applications of Theorem 3.2 in § 4 involve only two types of resources, we present an example (due to D. H. Wagner) where this theorem applies conveniently to allocate n resources in optimal fashion. Let β be a given function defined on R and $0 < \gamma_i < 1$, $i = 1, \cdots, n$, and $\sum_{i=1}^n \gamma_i < 1$. We wish to find nonnegative functions $g_1, \cdots, g_n$ on R which maximize

$$\int_R \beta(x) g_1(x)^{\gamma_1} \cdots g_n(x)^{\gamma_n} \, dx$$

subject to

$$\int_R [g_1(x) + \cdots + g_n(x)] \, dx \leqq 1.$$

By using Theorem 3.2, the solution is found to be

$$g_i^*(x) = \frac{\gamma_i \beta(x)^{1/(1-\sigma)}}{\sigma \int_R \beta(u)^{1/(1-\sigma)} \, du}, \qquad i = 1, \cdots, n,$$

where $\sigma = \sum_{j=1}^n \gamma_j$.

We also note that Theorem 3.2 has been applied by Persinger in [3] to find optimal search plans when one has the choice of searching with either of two sensors which cannot be used simultaneously.

4. Optimal search plans. In this section we consider two general search situations in which we may use the theorems of § 3 to find the optimal plan in Φ.

Throughout this section we let q' denote the derivative of q whenever q is a real-valued function defined on a subset of the real numbers. We shall understand $q(\infty)$ to be $\lim_{z \to \infty} q(z)$ whenever the limit exists.

4.1. Searches with interrupted contact investigation. In this subsection we use Theorem 3.1 to find the optimal search plan for the class of searches satisfying (2.1). For this class of searches the optimal plan calls for dividing effort between contact investigation and broad search. That is, instead of investigating a contact until it is identified as is required in [5], one must match each "increment" of broad search effort with a corresponding "increment" of contact investigation until the contact is identified.

For convenience, we list conditions (2.1) below:

(4.1)
(a) $\tilde{b} = b$ and b' exists, is positive, continuous, and strictly decreasing;

(b) A is continuous so that $\alpha'(w) = 1 - A(w)$ for $w \geqq 0$;

(c) a' exists and $\xi \equiv a'/(1 - A)$ is continuous, positive, and strictly decreasing;

(d) at least one of the following holds:

(i) the mean time to identify any false target is finite, i.e., $\lim_{w \to \infty} \alpha(w) < \infty$;
(ii) there exist $J > 0$ and $\varepsilon > 0$ such that $\zeta(w) \leqq J/w^{1+\varepsilon}$.

Mathematically our problem is to find $(m^*, \lambda^*) \in \Phi$ such that (1.11) is satisfied where **P** and **C** are given by (1.9) and (1.10), respectively, and

$$p(x, z, w) = f(x)b(z)a(w),$$

$$c(x, z, w) = \frac{z}{U} + \frac{\delta}{\Lambda} b(z)\alpha(w),$$

where $\alpha(w) = \int_0^w (1 - A(y))\, dy$. We then apply Proposition 1.1 to conclude that (m^*, λ^*) is optimal (i.e., minimizes the mean time to find the target among plans in Φ). Our method of attack is to find, for each $s > 0$, functions $(m^*(\cdot, s), \lambda^*(\cdot, s))$ and a multiplier $\gamma(s) > 0$ such that $(d^*, g^*) = (m^*(\cdot, s), \lambda^*(\cdot, s))$ and $k = \gamma(s)$ satisfy (i) through (iv) of Theorem 3.1. If we can do this in such a manner that $(m^*, \lambda^*) \in \Phi$, then (1.11) will hold and (m^*, λ^*) will be optimal. To this end, we compute, by virtue of (a) through (c) of (4.1),

$$\frac{D_2 p(x, z, w)}{D_2 c(x, z, w)} = \frac{\Lambda U f(x) b'(z) a(w)}{\Lambda + U\delta b'(z)\alpha(w)} \tag{4.2}$$

and

$$\frac{D_3 p(x, z, w)}{D_3 c(x, z, w)} = \frac{\Lambda f(x) a'(w)}{\delta(1 - A(w))} = \Lambda f(x)\frac{\zeta(w)}{\delta}. \tag{4.3}$$

Notice that condition (d) of Theorem 3.1 is satisfied. Conditions (a), (b), and (c) are easily checked with $\mathbf{U}(x) = \infty$ and $\mathbf{L}(x) = 0$.

We find functions u and v defined on $R \times (0, \infty)$ such that for each value of $k > 0$, $(d^*, g^*) = (u(\cdot, k), v(\cdot, k))$ satisfies (i) through (iv) of Theorem 3.1. By assumption (c) of (4.1), ζ^{-1}, the inverse function for ζ, exists on the domain $(\zeta(\infty), \zeta(0))$. For each $x \in R$ and $k > 0$, define

$$v(x, k) = \begin{cases} 0 & \text{if } k > \dfrac{\Lambda f(x)}{\delta}\zeta(0), \\ \zeta^{-1}\left(\dfrac{\delta k}{\Lambda f(x)}\right) & \text{if } \dfrac{\Lambda f(x)}{\delta}\zeta(\infty) < k \leqq \dfrac{\Lambda f(x)}{\delta}\zeta(0), \\ \infty & \text{if } k \leqq \dfrac{\Lambda f(x)}{\delta}\zeta(\infty). \end{cases} \tag{4.4}$$

Observe that if $\zeta(\infty) < k\delta/(\Lambda f(x)) \leqq \zeta(0)$, then $\Lambda f(x)\zeta(v(x,k))/\delta = k$, and by assumption (c),

$$D_3 p(x,z,w) \leqq kD_3 c(x,z,w) \quad \text{for } v(x,k) < w < \infty,$$
$$D_3 p(x,z,w) \geqq kD_3 c(x,z,w) \quad \text{for } 0 < w < v(x,k)$$

for all $0 < k < \infty$ and $z \geqq 0$. Also, $v(x,\cdot)$ is a decreasing, continuous function such that $\lim_{k\to 0} v(x,k) = \infty$.

Define, for $x \in R$ and $0 < k < \infty$,

$$r(x,k) = \frac{\Lambda k}{U[\Lambda f(x)a(v(x,k)) - k\delta\alpha(v(x,k))]}, \tag{4.5}$$
$$u(x,k) = \begin{cases} b'^{-1}(r(x,k)) & \text{if } 0 < r(x,k) \leqq b'(0), \\ 0 & \text{otherwise.} \end{cases}$$

Note that by condition (a), b'^{-1} is well-defined on $(0, b'(0)]$. When

$$0 < r(x,k) \leqq b'(0), \qquad D_2 p(x,u(x,k),v(x,k))/D_2 c(x,u(x,k),v(x,k)) = k.$$

It follows that for all $0 < k < \infty$, the decreasing nature of b' implies

$$D_2 p(x,z,v(x,k)) \leqq kD_2 c(x,z,v(x,k)) \quad \text{for } u(x,k) < z < \infty,$$
$$D_2 p(x,z,v(x,k)) \geqq kD_2 c(x,z,v(x,k)) \quad \text{for } 0 < w < u(x,k).$$

Thus, conditions (i) through (iv) of Theorem 3.1 are satisfied with $d^* = u(\cdot,k)$ and $g^* = v(\cdot,k)$. Therefore,

$$\mathbf{P}(u(\cdot,k),v(\cdot,k)) \leqq \mathbf{P}(d,g) \quad \text{implies} \quad \mathbf{C}(u(\cdot,k),v(\cdot,k)) \leqq \mathbf{C}(d,g) \quad \text{for } (d,g) \in G.$$

Note that each choice of k corresponds to an allocation of broad search effort requiring $H(k)$ broad search time to be executed, where

$$H(k) = \begin{cases} \int_R \frac{u(x,k)}{U}\,dx & \text{for } 0 < k < \infty, \\ 0 & \text{for } k = \infty, \\ \infty & \text{for } k = 0. \end{cases} \tag{4.6}$$

It now remains to find a function γ such that letting $m^*(\cdot,s) = u(\cdot,\gamma(s))$ and $\lambda^*(\cdot,s) = v(\cdot,\gamma(s))$ for $s > 0$, we obtain $\int_R (m^*(x,s)/U)\,dx = H(\gamma(s)) = s$ and $(m^*,\lambda^*) \in \Phi$. To do this, we state and prove a theorem which is an extension of Theorem 2 of [5].

THEOREM 4.1. *Let*

$$K_l = \sup\{k : H(k) = \infty\} \quad \textit{and} \quad K_u = \inf\{k : H(k) = 0\}.$$

Then under conditions (a) *through* (d) *of* (4.1), $\gamma = H^{-1}$, *the inverse of the function* H *restricted to* (K_l, K_u), *is well-defined, and*

$$m^*(x,s) \equiv u(x,\gamma(s)),$$
$$\lambda^*(x,s) \equiv v(x,\gamma(s)) \quad \textit{for } x \in R, \quad 0 \leqq s \leqq \infty,$$

are such that $(m^*, \lambda^*) \in \Phi$ *and* (m^*, λ^*) *minimizes the mean time to find the target among all plans in* Φ.

Proof. As in the proof of Theorem 2 of [5], we have $b'^{-1}(z) < 1/z$. Thus by (4.5),

$$H(k) = \int_R \frac{u(x,k)}{U}\,dx \leqq \int_{\{x:u(x,k)>0\}} \frac{f(x)}{k}\,dx \leqq \frac{1}{k} \quad \text{for } k > 0.$$

We prove that H^{-1} is well-defined by demonstrating that H is continuous, strictly decreasing, and that $H(k) \to \infty$ as $k \to 0$ and $H(k) \to 0$ as $k \to \infty$. We may proceed in the same manner as in Theorem 2 of [5] once we have shown that for each $x \in R$, $u(x, \cdot)$ is a decreasing function such that $\lim_{k\to 0} u(x,k) = \infty$. Fix $x \in R$. Note that b'^{-1} is decreasing and that $\lim_{z\to 0} b'^{-1}(z) = \infty$. Thus to demonstrate that $\lim_{k\to 0} u(x,k) = \infty$, it is sufficient to prove that $r(x,k) \downarrow 0$ as $k \downarrow 0$. To do this, it is sufficient to show that the function h given by

$$h(k) = \frac{\Lambda f(x)}{\delta}\frac{a(v(x,k))}{k} - \alpha(v(x,k)) \quad \text{for } 0 < k$$

is decreasing and $\lim_{k\to 0} h(k) = \infty$. To do this, we consider for $l > 0$,

$$\begin{aligned} h(k+l) - h(k) &= \frac{\Lambda f(x)}{\delta}\left[\frac{a(v(x,k+l))}{k+l} - \frac{a(v(x,k))}{k}\right] - [\alpha(v(x,k+l)) - \alpha(v(x,k))] \\ &\leqq \frac{\Lambda f(x)}{\delta}\frac{a(v(x,k+l)) - a(v(x,k))}{k} - [\alpha(v(x,k+l)) - \alpha(v(x,k))]. \end{aligned}$$

If $k \geqq k_0 = \Lambda f(x)a'(0)/(\delta(1 - A(0)))$, then $v(x,k) = v(x,k+l) = 0$ and $h(k+l) - h(k) \leqq 0$. If $k < k_0$, then condition (c) of (4.1) and the nonincreasing nature of $v(x,\cdot)$ guarantee that

$$a'(v(x,y)) \leqq \frac{k\delta}{\Lambda f(x)}(1 - A(v(x,y))) \quad \text{for } k_0 > y > k.$$

Since $v(x,\cdot)$ is continuous and monotone nonincreasing, we have for small enough l,

$$\begin{aligned} a(v(x,k+l)) - a(v(x,k)) &= \int_k^{k+l} a'(v(x,y))v(x,dy) \\ &\leqq \frac{k\delta}{\Lambda f(x)}\int_k^{k+l} [1 - A(v(x,y))]v(x,dy) \\ &= \frac{k\delta}{\Lambda f(x)}[\alpha(v(x,k+l)) - \alpha(v(x,k))]. \end{aligned}$$

From this, it follows that $h(k+l) - h(k) \leqq 0$.

That $\lim_{k\to 0} h(k) = \infty$ follows easily from the nonincreasing nature of $v(x,\cdot)$ and (i) of assumption (d). To see that condition (ii) of (d) also implies $\lim_{k\to 0} h(k) = \infty$, we note that this condition gives $\zeta^{-1}(z) \leqq (J/z)^{1/(1+\varepsilon)}$ for sufficiently small z

and write for sufficiently small k,

$$h(k) \geqq \frac{\Lambda f(x)}{\delta k}\left[a(v(x,k)) - \frac{\delta k}{\Lambda f(x)}\zeta^{-1}\left(\frac{\delta k}{\Lambda f(x)}\right)\right]$$
$$\geqq \frac{\Lambda f(x)}{\delta k}\left[a(v(x,k)) - J^{1/(1+\varepsilon)}\left(\frac{\delta k}{\Lambda f(x)}\right)^{\varepsilon/(1+\varepsilon)}\right].$$

From this, it is clear that $\lim_{k\to 0} h(k) = \infty$.

The proof of the fact that by restricting the domain of H to (K_l, K_u) we may define H^{-1} such that $H(H^{-1}(s)) = s$ for $s \geqq 0$ now follows in the same manner as in Theorem 2 of [5].

To check that (m^*, λ^*) is in Φ, it is sufficient to observe that the function γ defined by $\gamma(s) = H^{-1}(s)$ for $s \geqq 0$ is continuous and decreasing. From this observation, it follows that $m^*(x, \cdot)$ and $\lambda^*(x, \cdot)$ are continuous and increasing. By the definition of H^{-1},

$$\int_R \frac{m^*(x,s)}{U}\,dx = \int_R \frac{u(x,\gamma(s))}{U}\,dx = s \quad \text{for } s \geqq 0.$$

Thus, $(m^*, \lambda^*) \in \Phi$.

One may check that for each $s \geqq 0$, the conditions of Theorem 3.1 are satisfied with $k = \gamma(s)$, $d^*(x) = m^*(x, s)$, and $g^*(x) = \lambda^*(x, s)$. Thus for any d and g in G and $s \geqq 0$,

$$\text{(4.7)} \qquad \mathbf{P}(d,g) \geqq \mathbf{P}(m^*(\cdot,s), \lambda^*(\cdot,s)) \quad \text{implies} \quad \mathbf{C}(d,g) \geqq \mathbf{C}(m^*(\cdot,s), \lambda^*(\cdot,s)).$$

Since $m(x,\cdot)$ and $\lambda(x,\cdot)$ are strictly increasing when $m(x,\cdot) > 0$ and $\lambda(x,\cdot) > 0$, one may check that $P(m^*, \lambda^*, \cdot)$ is strictly increasing. Continuity of $P(m^*, \lambda^*, \cdot)$ follows easily from the continuity of $m^*(x,\cdot)$ and $\lambda^*(x,\cdot)$ and the monotone convergence theorem. Thus, by Proposition 1.1, (m^*, λ^*) minimizes the mean time to find the target among plans in Φ. This concludes the proof.

We consider two search examples in which the optimal plan may be found by Theorem 4.1.

For the first example, we assume

$$\tilde{b}(z) = b(z) = 1 - e^{-z} \quad \text{for } z \geqq 0,$$
$$a(w) = 1 - e^{-w} \quad \text{for } w \geqq 0,$$
$$A(w) = 0 \quad \text{for } w \geqq 0.$$

That is, it is not possible to identify false targets. Only the contact which is the target may be identified. Thus, $\alpha(w) = w$ for $w \geqq 0$. It follows that $\zeta(w) = e^{-w}$ so that conditions (a) through (c) and condition (ii) under (d) in (4.1) are satisfied.

In particular, we consider the case where the target location density is given (in polar coordinates) by

$$\text{(4.8)} \qquad f(r,\theta) = \frac{1}{2\pi\sigma^2}\exp\left(-\frac{r^2}{2\sigma^2}\right) \quad \text{for } r \geqq 0, \quad 0 \leqq \theta \leqq 2\pi.$$

Since the density does not depend on θ, we shall write $f(r)$ for $f(r,\theta)$. We shall use the same convention concerning all functions defined on the plane which do not depend on θ.

For convenience, we define τ to be the unique positive solution of $\tau - \ln(\tau) - 1 = \Lambda/(\delta U)$. Note that $\tau \geqq 1$.

The conditions of Theorem 4.1 are satisfied and the functions v and u become

$$v(r,k) = \max\left\{0, \ln\left(\frac{\Lambda f(r)}{\delta k}\right)\right\} \qquad \text{for } 0 < k, \quad r \geqq 0,$$

$$u(r,k) = \max\left\{0, \ln\left[\frac{U\delta}{\Lambda}\left(\frac{\Lambda f(r)}{\delta k} - 1 - \ln\left(\frac{\Lambda f(r)}{\delta k}\right)\right)\right]\right\} \qquad \text{for } 0 < k, \quad r \geqq 0.$$

Thus,

$$H(k) = \frac{2\pi}{U}\int_0^{L(k)} \ln\left(A_1 e^{-r^2/2\sigma^2} + A_2 r^2 + A_3\right) r\,dr \quad \text{for } 0 < k < \frac{\rho}{\tau},$$

where

$$L(k) = \sigma\left[2\ln\left(\frac{\rho}{\tau k}\right)\right]^{1/2},$$

$$A_1 = \frac{U}{2\pi\sigma^2 k}, \quad A_2 = \frac{\delta U}{2\sigma^2\Lambda}, \quad A_3 = -\frac{\delta U}{\Lambda}\left(\ln\left(\frac{\rho}{k}\right) + 1\right), \quad \text{and } \rho = \frac{\Lambda}{2\pi\delta\sigma^2}.$$

Letting $\gamma(s) = H^{-1}(s)$ for $s \geqq 0$, we have as the optimal plan

$$m^*(r,s) = \begin{cases} \ln\left(\dfrac{U\delta}{\Lambda}\left[\dfrac{\rho\, e^{-r^2/2\sigma^2}}{\gamma(s)} - 1 - \ln\left(\dfrac{\rho\, e^{-r^2/2\sigma^2}}{\gamma(s)}\right)\right]\right) & \\ \qquad\qquad \text{for } 0 \leqq r \leqq \sigma\left[2\ln\left(\dfrac{\rho}{\tau\gamma(s)}\right)\right]^{1/2}, & \\ 0 \qquad\qquad \text{otherwise}, & \end{cases}$$

$$\lambda^*(r,s) = \begin{cases} \ln\left(\dfrac{\rho}{\gamma(s)}\right) - \dfrac{r^2}{2\sigma^2} & \text{for } 0 \leqq r \leqq \left(\dfrac{\ln(\rho/\gamma(s))}{2\sigma^2}\right)^{1/2}, \\ 0 & \text{otherwise}. \end{cases}$$

As an example of a search which satisfies condition (i) of (d), we take f, $\tilde{b}$, b, and a to be the same as above but define $A(w) = 1 - e^{-w} - w\,e^{-w}$ for $w \geqq 0$. Then one may check that $\alpha(w) = 2(1 - e^{-w}) - w\,e^{-w}$ for $w \geqq 0$ and that conditions (a) through (c) and (i) under (d) of (4.1) are satisfied. Thus, Theorem 4.1 is applicable and (m^*, λ^*) may be found in a manner similar to the above.

4.2. Searches for which contact investigation is immediate and uninterrupted. For the class of searches satisfying (2.2), we find the optimal plan $(m^*, \lambda^*) \in \Phi$ and show that it calls for contacts to be investigated immediately, and that once a contact investigation is begun, it continues until the contact is identified. By virtue of Proposition 1.1, our problem is to find $(m^*, \lambda^*) \in \Phi$ such that $(m^*(\cdot, s), \lambda^*(\cdot, s))$ satisfies (1.11) for $s > 0$, where $p, c, \mathbf{P}$, and $\mathbf{C}$ are given by (1.7) through

(1.10). For convenience, we restate (2.2) below:

(a) both b and $\tilde{b}$ have derivatives which are positive, continuous, and strictly decreasing;

(4.9) (b) $\tilde{b}/b$ and $\tilde{b}'/b'$ are nondecreasing on $(0, \infty)$; and

(c) α/a is nonincreasing on $(0, \infty)$.

Since $p(x, z, w) = f(x)b(z)a(w)$, we may write

$$c(x, z, w) = \frac{z}{U} + \frac{\delta}{\Lambda f(x)} \frac{\tilde{b}(z)}{b(z)} \frac{\alpha(w)}{a(w)} p(x, z, w) \tag{4.10}$$

when $p(x, z, w) > 0$. Suppose $p(x, z, w) = p_0 > 0$. Then there is a z_0 such that $p(x, z_0, \infty) = p_0$. From (4.10) and (b) and (c) of (4.9), it is clear that

$$c(x, z_0, \infty) = \min \{c(x, z, w) : p(x, z, w) = p_0\}.$$

This leads one to conjecture that contact investigation should be immediate and uninterrupted (i.e., $\lambda^*(x, s) = \infty$ for $x \in R$ and $s > 0$) and that one may reduce the problem of finding the optimal plan to that of finding m^* to minimize the mean time to contact the target, where the probability of detecting the target by broad search time s is given by

$$\hat{P}(m, s) = \int_R \hat{p}(x, m(x, s))\, dx$$

where

$$\hat{p}(x, z) = p(x, z, \infty) = f(x)b(z),$$

and the expected cost (in time) to achieve detection probability $\hat{P}(m, s)$ is given by

$$\hat{C}(m, s) = \int_R \hat{c}(x, m(x, s))\, dx$$

where

$$\hat{c}(x, z) = c(x, z, \infty) = \frac{z}{U} + \frac{\delta\alpha(\infty)}{\Lambda} \tilde{b}(z).$$

Note that (c) of (4.9) guarantees that $\alpha(\infty) < \infty$. By the use of Corollary 3.2, Theorem 4.2 below shows that this conjecture is true.

In order to state Theorem 4.2, we define

$$\hat{r}(x, z) = \frac{D_2\hat{p}(x, z)}{D_2\hat{c}(x, z)} = \frac{f(x)b'(z)}{1/U + (\delta\alpha(\infty)/\Lambda)\tilde{b}'(z)}. \tag{4.11}$$

Since (4.9) holds, we have that $\hat{r}(x, \cdot)$ is continuous and strictly decreasing. In view of this, $\hat{r}^{-1}(x, \cdot)$ is well-defined on $[\hat{r}(x, 0), \hat{r}(x, \infty)]$. For convenience of notation, we define $\hat{r}^{-1}(x, k) = 0$ for $k > \hat{r}(x, 0)$ and $\hat{r}^{-1}(x, k) = \infty$ for $k < \hat{r}(x, \infty)$. Define

$$H(k) = \int_R \frac{\hat{r}^{-1}(x, k)}{U}\, dx \quad \text{for } 0 \leqq k \leqq \infty. \tag{4.12}$$

Note that $H(0) = \infty$ and $H(\infty) = 0$. We now state Theorem 4.2.

THEOREM 4.2. *Suppose that* (4.9) *holds. Then the optimal plan* $(m^*, \lambda^*) \in \Phi$ *requires that contact investigation be immediate and uninterrupted until the contact is identified (i.e.,* $\lambda^*(x, s) = \infty$ *for* $x \in R$ *and* $s > 0$*). Moreover,* γ*, the inverse of* H *restricted to* (K_l, K_u) *where* $K_l = \sup\{k: H(k) = \infty\}$ *and* $K_u = \inf\{k: H(k) = 0\}$*, is well-defined and*

$$m^*(x, s) = \hat{r}^{-1}(x, \gamma(s)) \quad \text{for } x \in R \text{ and } s > 0.$$

Proof. Observe that

$$Uf(x) \geqq Uf(x) \int_0^z b'(y)\, dy \geqq \int_0^z \hat{r}(x, y)\, dy \geqq z\hat{r}(x, z),$$

where the last inequality follows from the decreasing nature of $\hat{r}(x, \cdot)$. Thus, $\hat{r}(x, z) \leqq Uf(x)/z$ which implies $\hat{r}^{-1}(x, z) \leqq Uf(x)/z$. Hence,

$$H(k) = \int_R \frac{\hat{r}^{-1}(x, k)}{U}\, dx \leqq \int_{\{x: \hat{r}^{-1}(x,k) > 0\}} \frac{\hat{r}^{-1}(x, k)}{U}\, dx$$
$$\leqq \int_R \frac{f(x)}{k}\, dx = \frac{1}{k},$$

and we may show that H, restricted to (K_l, K_u), is invertible in the same manner as in Theorem 2 of [5]. Thus, γ is well-defined on $(0, \infty)$.

We now verify that for each $s > 0$, $k = \gamma(s)$ and $(d^*, g^*) = (m^*(\cdot, s), \infty)$ satisfy the conditions of Corollary 3.2 with $e = p$, $E = \mathbf{P}$, $\mathbf{L} = 0$, and $\mathbf{U} = \infty$. Observe that $\Delta(x) = [0, f(x)]$ and that $\varphi_1(x, q) = b^{-1}(q/f(x))$ is differentiable and satisfies

$$p(x, \varphi_1(x, q), \infty) = q.$$

Moreover, by the discussion preceding this theorem,

$$\xi(x, q) = c(x, \varphi_1(x, q), \infty) = \min\{c(x, z, w): p(x, z, w) = q\}.$$

It is easily checked that $\varphi_1(x, \cdot)$ is strictly increasing and that $\xi'(x, \cdot)$ exists and is Riemann integrable. Also, $\hat{p}(x, \cdot)$, as defined above, is strictly increasing. Since

$$D_2\hat{p}(x, \hat{r}^{-1}(x, k))/D_2\hat{c}(x, \hat{r}^{-1}(x, k)) = k$$

and since $\hat{r}(x, \cdot)$ is decreasing, it is clear that $k = \gamma(s)$ and $d^* = m^*(\cdot, s)$ satisfy (3.9) and (3.10). Thus, Corollary 3.2 yields that $(m^*(\cdot, s), \lambda^*(\cdot, s)) = (\hat{r}^{-1}(\cdot, \gamma(s)), \infty)$ satisfies (1.11) for all $s > 0$. It is easy to verify that $(m^*, \lambda^*) \in \Phi$ and that $P(m^*, \lambda^*, \cdot)$ is continuous and strictly increasing. Thus, by Proposition 1.1, (m^*, λ^*) is optimal in Φ. This proves the theorem.

As an example, consider a search in which, fixing $\Omega > \omega > 0$,

$$b(z) = 1 - e^{-\Omega z}, \quad \tilde{b}(z) = 1 - e^{-\omega z}, \quad z \geqq 0,$$
$$a(w) = A(w) = 1 - e^{-w}, \quad w \geqq 0,$$

and δ, Λ, and U are positive constants. Then $\alpha(w)/a(w) = 1$ for $w > 0$, and one can check that (4.9) holds. Consider the case $\Omega = 2\omega$. Let

$$\eta(x, k) = \frac{-Uk\delta\omega + \{(Uk\delta\omega)^2 + 8\omega U\Lambda^2 kf(x)\}^{1/2}}{2\Lambda k} \quad \text{for } k > 0 \text{ and } x \in R.$$

Then the function $\hat{r}^{-1}$ becomes

$$\hat{r}^{-1}(x, k) = \begin{cases} \dfrac{1}{\omega} \ln(\eta(x, k)) & \text{for } 1 \geqq \dfrac{1}{\eta(x, k)} > 0, \\ 0 & \text{otherwise.} \end{cases}$$

If f is given by (4.8), it can be shown that H in (4.12) is

$$H(k) = \frac{2\pi\sigma^2}{U\omega} \ln\left(\frac{U\delta\omega}{2\Lambda}\right) \ln\left(\frac{\omega U\Lambda}{\pi\sigma^2 k(\Lambda + U\delta\omega)}\right) - \frac{2\pi\sigma^2}{\omega} \int_{l_1}^{l_2} \frac{\ln(-1 + \sqrt{1 + x})\,dx}{x},$$

where

$$l_1 = \frac{4\Lambda}{\delta\omega U}\left(\frac{\Lambda}{\pi\sigma^2 k\sigma}\right) \quad \text{and} \quad l_2 = \frac{4\Lambda}{\delta\omega U}\left(1 + \frac{\Lambda}{\delta\omega U}\right).$$

Numerical methods must now be used to evaluate and invert H to obtain $\gamma(s) = H^{-1}(s)$. The optimal plan is then given by

$$m^*(x, s) = \hat{r}^{-1}(x, \gamma(s)) \quad \text{and} \quad \lambda^*(x, s) = \infty \quad \text{for } x \in R \text{ and } s > 0.$$

5. Optimal search using a sensor with a stochastic sweep width. In this section we consider the problem of optimal search using a sensor with a stochastic, non-time-varying sweep width having a known prior distribution. We show how this problem may be handled by the methods of § 4. In an example, we find the optimal search plan for a search involving a stochastic sweep width and false targets. Specializing this example to allow no false targets (i.e., $\delta = 0$), we are able to calculate the mean time to find the target using the optimal plan.

Uncertainty in the sweep width of search sensors is a common problem in searches. This is usually due to insufficient testing of the search equipment against objects similar to the target or in environments similar to that of the search region. However, the problem may arise even when there is very good knowledge of the sensor's capabilities, particularly in underwater search. For example, it may not be known whether the target is in a single piece or many small ones. There may be doubt as to whether the target is buried in mud or sitting on top of a sandy bottom. If for each possible target condition the sweep width of the search sensor were known, then by assigning probabilities to each target condition, one could generate a prior distribution for sweep width. Further results concerning stochastic sweep width may be found in [4].

Although we shall characterize the effectiveness of a sensor by its sweep width, it is clear that other parameters which characterize the effectiveness may be used in place of sweep width in the discussion below.

Let $\mathbf{\Omega}$ be the random variable giving the sweep width of the broad search sensor. Note that we are assuming that the value of the sweep width, although unknown, is fixed throughout the search. We suppose that there are functions

B and $\tilde{B}$ defined on $R \times [0, \infty) \times [0, \infty)$ such that $B(x, z, \Omega)$ is the probability of contacting the target with z amount of effort given that $\boldsymbol{\Omega} = \Omega$ and that the target is located at x and such that $\tilde{B}(x, z, \Omega)$ gives the corresponding probability for contacting false targets. Moreover, we assume that given $\boldsymbol{\Omega} = \Omega$, the search assumptions of § 1 hold with b and $\tilde{b}$ replaced by $B(\cdot, \cdot, \Omega)$ and $\tilde{B}(\cdot, \cdot, \Omega)$ respectively.

Let $L(\Omega) = \Pr\{\boldsymbol{\Omega} \leqq \Omega\}$. We compute

$$b(x, z) = \int_0^\infty B(x, z, \Omega)L(d\Omega),$$

$$\tilde{b}(x, z) = \int_0^\infty \tilde{B}(x, z, \Omega)L(d\Omega).$$

The problem of optimal search with a stochastic sweep width is then reduced to the optimal search problem considered in this paper with b and $\tilde{b}$ as computed above.

As an example, we find the optimal search plan (m^*, λ^*) in Φ for a search involving a stochastic sweep width and false targets. For this example, the false target density δ is constant and the target location distribution is circular normal with density function f given by

$$f(x) = \frac{1}{2\pi\sigma^2} \exp\left[-\frac{(x_1^2 + x_2^2)}{2\sigma^2} \right] \quad \text{for } x = (x_1, x_2) \in R, \tag{5.1}$$

where $\sigma > 0$. We suppose that

$$\tilde{B}(x, z, \Omega) = B(x, z, \Omega) = 1 - e^{-\Omega z} \quad \text{for } x \in R, \quad z \geqq 0, \quad \Omega > 0,$$

and that $\boldsymbol{\Omega}$ is gamma distributed with density

$$l(\Omega) = \frac{\beta^\nu \Omega^{\nu - 1} e^{-\beta\Omega}}{\Gamma(\nu)} \quad \text{for } \Omega > 0,$$

where β and ν are positive parameters and Γ is the well-known gamma function. Then for all $x \in R$,

$$\tilde{b}(x, z) = b(x, z) = 1 - \beta^\nu/(\beta + z)^\nu \quad \text{for } z \geqq 0.$$

We assume that

$$a(x, w) = A(x, w) = 1 - e^{-w/\eta} \quad \text{for } w > 0 \text{ and } x \in R,$$

where η is a positive constant. Hence, $\alpha(x, w) = \eta(1 - e^{-w/\eta})$. We take the contact investigation and broad search rates Λ and U to be positive constants so that the mean time to identify a contact is given by

$$T = \alpha(x, \infty)/\Lambda = \eta/\Lambda.$$

One may easily verify that conditions (a) through (c) of (4.9) are satisfied by this example. Thus, by Theorem 4.2, the optimal plan calls for contact investigation to be immediate and uninterrupted until the contact is identified (i.e.,

$\lambda^*(x, s) = \infty$). Moreover, the functions $\hat{p}$, $\hat{c}$ and $\hat{r}$ of § 4.2 become

$$\hat{p}(x, z) = f(x)\left[1 - \left(\frac{\beta}{\beta + z}\right)^{\nu}\right],$$

$$\hat{c}(x, z) = \frac{z}{U} + \delta T\left[1 - \left(\frac{\beta}{\beta + z}\right)^{\nu}\right],$$

$$\hat{r}(x, z) = \frac{\nu\beta^{\nu} f(x)}{(\beta + z)^{\nu+1}/U + \delta T \nu \beta^{\nu}} \quad \text{for } x \in R, \quad z \geqq 0.$$

Thus, by Theorem 4.2,

(5.2) $$m^*(x, s) = \hat{r}^{-1}(x, \gamma(s)) \quad \text{and} \quad \lambda^*(x, s) = \infty \quad \text{for } x \in R \text{ and } s \geqq 0,$$

where γ is the inverse of H defined in (4.11).

As a special case, we take $\delta = 0$ (i.e., no false targets). For this case, we write $\mu(m^*)$ and $P(m^*, s)$ in place of $\mu(m^*, \infty)$ and $P(m^*, \infty, s)$ respectively. Since $\delta = 0$, $\mu(m^*)$ gives the mean time to find the target when using plan m^*. We now calculate $\mu(m^*)$ as follows:

(5.3) $$\hat{r}^{-1}(x, k) = \max\left\{0, \beta\left[\left(\frac{\nu f(x)}{\beta k}\right)^{1/(\nu+1)} - 1\right]\right\} \quad \text{for } k > 0$$

and

$$H(k) = \frac{\nu(\nu + 1)}{U}\xi\left[(\xi k)^{-1/(\nu+1)} + \frac{\ln(\xi k)}{\nu + 1} - 1\right] \quad \text{for } 0 < k < \frac{1}{\xi},$$

where $\xi = 2\pi\sigma^2\beta/\nu$. Using (5.1) through (5.3), one can calculate

$$P(m^*, s) = 1 - (\nu + 1)[\xi\gamma(s)]^{\nu/(\nu+1)} + \nu\xi\gamma(s) \quad \text{for } s > 0.$$

Observe that

$$\mu(m^*) = \int_0^{\infty} [1 - P(m^*, s)]\, ds.$$

Since $s = H(\gamma(s))$, we may write

(5.4) $$\mu^* = \int_{1/\xi}^{0} [1 - P(m^*, H(k))]H'(k)\, dk = \begin{cases} \dfrac{2\pi\sigma^2\beta}{U}\dfrac{3\nu + 1}{\nu(\nu - 1)} & \text{for } \nu > 1, \\ \infty & \text{for } 0 < \nu \leqq 1. \end{cases}$$

6. Remarks. The first and third authors, who have prepared this paper, wish to explicitly identify the central contributions of the second author, made largely as a graduate student on summer work. In particular, he is responsible for Theorem 3.1 and for the seminal idea which resulted in Theorem 3.2. He also found the optimal search plan which now appears as the first example of § 4.1. From this initial example of optimally allocating two types of search effort, the rest of this paper eventually evolved.

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