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OPTIMIZATION OF ALLOCATIONS UNDER A COVERABILITY CONDITION*

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Abstract. Consider maximization of a real functional E given by $E(q) = \int_X e(x, q(x))\, d\mu x$, subject to equality or inequality constraint on $\int_X q\, d\mu$. It is proved that such extrema exist and necessitate satisfaction of a pointwise multiplier rule without assuming a topology on X, but assuming a condition called coverability of e, pertaining to the concave envelope of $e(x, \cdot)$, $x \in X$. Examples show that key hypotheses may not be omitted.

This paper relates to the results in [6] on the necessity of a pointwise multiplier rule for constrained maximization of a separable functional and on existence of such extrema. A simple constraint functional is used, but the objective functional is subjected to a "coverability" condition defined below in terms of concave envelopes. No Borel assumptions are made, in contrast to Corollary 5.2 and Theorem 6.13 of [6]; in further contrast to the latter, our assumption below differs from $Z(x)$ being upper closed. Necessity results are given as Theorem 5 and Corollary 6, and existence is given in Theorem 8. Remarks 7 and 9 show by examples that the results cannot be strengthened in various ways.

The usages below are consistent with [6], and are in some ways simpler. We fix an arbitrary nonvacuous set X (on which no topology is assumed). For $x \in X$, let $Y(x) \neq \emptyset$ be a real interval (not necessarily bounded or closed). For $x \in X$, define $T(x) = \inf Y(x)$ and $U(x) = \sup Y(x)$. Assume T and U are measurable (extended real-valued) functions. Let $\Omega = \{(x, y): x \in X \text{ and } y \in Y(x)\}$. Fix a measure μ over X, to which measurability and integrability refer unless stated otherwise. We ignore subsets of X having μ measure 0, e.g., "for $x \in X$" means "for μ a.e. $x \in X$." Denoting one-dimensional Lebesgue measure by $\mathscr{L}$, we use the product measure $\mu \times \mathscr{L}$ on Ω.

Let ω be the positive integers and $\mathscr{E}_1$ be the reals. Fixing $e: \Omega \to \mathscr{E}_1$, define

$$\Xi = \{q: q(x) \in Y(x) \text{ for } x \in X \text{ and } e(\cdot, q(\cdot)) \text{ and } q \text{ are measurable}\},$$

$$\Phi = \Xi \cap \{q: e(\cdot, q(\cdot)) \text{ and } q \text{ are integrable}\},$$

$$E(q) = \int_X e(x, q(x))\, d\mu x \quad \text{and} \quad C(q) = \int_X q\, d\mu \quad \text{for } q \in \Phi.$$

We say $q^* \in \Phi$ is optimal (strongly optimal) if

$$E(q^*) = \max\{E(q): C(q) = C(q^*)\} \quad (E(q^*) = \max\{E(q): C(q) \leqq C(q^*)\}).$$

Let $\lambda \in \mathscr{E}_1$. Define $l_\lambda(x, y) = e(x, y) - \lambda y$ for $(x, y) \in \Omega$. For $q \in \Phi$, if $\lambda = 0$ and $C(q) = \pm\infty$, define $L_\lambda(q) = E(q)$; otherwise, define $L_\lambda(q) = E(q) - \lambda C(q)$ if this exists. For $q^* \in \Phi$, we say (q^*, λ) *satisfies the functional multiplier rule* if

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$L_\lambda(q^*) \geqq L_\lambda(q)$ whenever $L_\lambda(q)$ exists; (q^*, λ) *satisfies the pointwise multiplier rule* if $l_\lambda(x, q^*(x)) \geqq l_\lambda(x, y)$ for $y \in Y(x)$, $x \in X$. If either rule is satisfied and $\lambda \geqq 0$, we say it is *strongly* satisfied.

Differentiation is always with respect to the last component of the argument and is always one-sided. A superscript + or − on a function denotes right or left derivative respectively, e.g., $e^+(x, y) = \lim_{\delta \downarrow 0}[e(x, y + \delta) - e(x, y)]/\delta$.

Suppose F is a real interval, $f: F \to \mathscr{E}_1$, and for $t, u \in F$ and $0 \leqq \alpha \leqq 1$, $f(\alpha t + (1 - \alpha)u) \geqq \alpha f(t) + (1 - \alpha)f(u)$. We then say f is *concave*. On interior F, f is continuous, f^+ and f^- both exist, and except on a countable set $f^+ = f^-$; at endpoints of F, if any, f might be discontinuous. We say $y \in F$ is an *extremizing point* of f if there are no $z, w \in F$ and $0 < \beta < 1$ such that $z < w$ and $(y, f(y)) = \beta(z, f(z)) + (1 - \beta)(w, f(w))$.

If h and g are real-valued functions on F, we say that g is the *concave envelope* of h if (a) g is concave and continuous, (b) $g(y) \geqq h(y)$ for $y \in F$, and (c) whenever $\hat{g}$ is a continuous concave function on F such that $\hat{g}(y) \geqq h(y)$ for $y \in F$, we have $g(y) \leqq \hat{g}(y)$ for $y \in F$.

Assumption 1. Throughout we assume that e is *covered* by m, i.e., e is *coverable*, meaning that the following conditions are satisfied:

(i) Ω is $\mu \times \mathscr{L}$ measurable and $m: \Omega \to \mathscr{E}_1$ is a $\mu \times \mathscr{L}$ measurable function;
(ii) for $x \in X$, $m(x, \cdot)$ is the concave envelope of $e(x, \cdot)$;
(iii) for $x \in X$, $m(x, y) = e(x, y)$ whenever y is an extremizing point of $m(x, \cdot)$ (equivalently, $e(x, \cdot)$ is upper semicontinuous at each such y);
(iv) for $x \in X$ and $y \in Y(x)$, there exist extremizing points w and z of $m(x, \cdot)$ such that $w \leqq y \leqq z$.

For $(x, y) \in \Omega$, we define

$$\psi(x, y) = \inf\{z : z \geqq y \text{ and } z \text{ is an extremizing point of } m(x, \cdot)\}.$$

The following lemma, stated without proof, gives equivalent conditions on a point b, extendable to a function q on X by regarding $b = q(x)$ for a particular $x \in X$. As such, they provide alternative formulations of the pointwise multiplier rule; (iii) corresponds to a functional Neyman–Pearson condition as in [7].

LEMMA 2. *Let F be a real interval, $h: F \to \mathscr{E}_1$, g be the concave envelope of h, $b \in F$, and $\lambda \in \mathscr{E}_1$. Then the following six conditions are equivalent*:

(i) $h(b) - \lambda b = \max\{h(y) - \lambda y : y \in F\}$;
(ii) $h(b) = g(b)$ *and* $g(b) - \lambda b = \max\{g(y) - \lambda y : y \in F\}$;
(iii) $h(b) = g(b)$, $g^+(y) \geqq \lambda$ *for* $b > y \in F$, *and* $g^-(y) \leqq \lambda$ *for* $b < y \in F$;
(iv) $h(b) = g(b)$, $g^+(b) \leqq \lambda$ *if* $b < \sup F$, *and* $\lambda \leqq g^-(b)$ *if* $b > \inf F$;
(v) $h(b) = g(b)$ *and not* $[g^+(b) > \lambda$ *or* $g^-(b) < \lambda]$;
(vi) *the graph of h lies on or below the line through $(b, h(b))$ with slope λ.*

LEMMA 3. *For $(x, y) \in \Omega$, (i) $\psi(x, y) \in Y(x)$ and (ii) $\psi(x, y)$ is an extremizing point of $m(x, \cdot)$.*

Proof. From (iv) in Assumption 1, we obtain (i). If $\psi(x, y)$ were not an extremizing point of $m(x, \cdot)$, we could obtain $z, w \in Y(x)$ with $z < \psi(x, y) < w$ and $(\psi(x, y), m(x, \psi(x, y)))$ lying on the chord joining $(z, m(x, z))$ and $(w, m(x, w))$; but the interior of this chord must contain extremizing points of $m(x, \cdot)$ (by definition of ψ), in contradiction. Hence (ii) holds.

LEMMA 4. *Suppose $P \subset X$ is measurable, p is a measurable function on P, and $p(x) \in Y(x)$ for $x \in P$. Then:*

(i) *if we define $g(x, y) = p(x)$ for $x \in P$ and $y \in Y(x)$, then g is a $\mu \times \mathscr{L}$ measurable function;*

(ii) *if $q: P \to \mathscr{E}_1$, $A \equiv (P \times \mathscr{E}_1) \cap \{(x, y): y \leqq q(x)\}$ is $\mu \times \mathscr{L}$ measurable, and μ is σ-finite on P, then q is a measurable function;*

(iii) *m^+ and m^- are $\mu \times \mathscr{L}$ measurable functions;*

(iv) *if μ is σ-finite on P, then $m(\cdot, p(\cdot))$, $m^+(\cdot, p(\cdot))$, $m^-(\cdot, p(\cdot))$, and $\psi(\cdot, p(\cdot))$ are measurable functions.*

Proof. To prove (i), note that for $a \in \mathscr{E}_1$,

$$\{(x, y): g(x, y) > a\} = \Omega \cap [\{x: p(x) > a\} \times \mathscr{E}_1],$$

which is $\mu \times \mathscr{L}$ measurable. To prove (iii), define $g_0(x, y) = U(x), x \in X$. Then m^+ is defined on

$$\Omega \cap \{(x, y): y < U(x)\} = \Omega - (X \times \mathscr{E}_1) \cap \{(x, y): g_0(x, y) - y = 0\}$$

which is $\mu \times \mathscr{L}$ measurable by (i). Let $\delta \in \mathscr{E}_1$, and for $(x, y) \in \Omega$, let $\tau_\delta(x, y) = (x, y - \delta)$ and $m_\delta(x, y - \delta) = m(x, y)$. Then $\mu \times \mathscr{L}$ is invariant under τ_δ and for $a \in \mathscr{E}_1$, $\{(x, y): m_\delta(x, y) \geqq a\} = \tau_\delta(\{(x, y): m(x, y) \geqq a\})$. Thus, m_δ is $\mu \times \mathscr{L}$ measurable, so is m^+, being a limit of such, and similarly so is m^-, proving (iii).

Proof of (ii) primarily follows [3, exercises (5e) and (5f), § 34]. Suppose $a \in \mathscr{E}_1$. One may show that (under the measure foundations of [2] or [3]) since A is $\mu \times \mathscr{L}$ measurable, so is $D_n \equiv \{(x, y): (x, a + y/n) \in A\}$ for $n \in \omega$. Let $B = P \cap \{x: q(x) > a\}$. Then

$$B \times (0, 1] = \bigcup_{n=1}^{\infty} \{(x, y): x \in P, a + y/n \leqq q(x), \text{and } 0 < y \leqq 1\}$$

$$= \bigcup_{n=1}^{\infty} [D_n \cap (P \times (0, 1])].$$

Hence, $B \times (0, 1]$ is $\mu \times \mathscr{L}$ measurable. With μ σ-finite on P, Fubini's theorem applied to the indicator function of $B \times (0, 1]$ shows B is measurable, proving (ii).

To prove (iv), let $b \in \mathscr{E}_1$, $K = \{(x, y): m(x, y) > b \text{ and } x \in P\}$, and

$$r(x) = \inf\{y: (x, y) \in K \text{ or } m^-(x, y) < 0\},$$

$$s(x) = \sup\{y: (x, y) \in K \text{ or } m^+(x, y) > 0\} \quad \text{for } x \in P.$$

Since $m(x, \cdot)$ is concave and continuous for $x \in X$,

$$(P \times \mathscr{E}_1) \cap \{(x, y): y \leqq r(x)\}$$
$$= (P \times \mathscr{E}_1) \cap \{(x, y): [m(x, y) \leqq b \text{ and } m^-(x, y) \geqq 0] \text{ or } y \leqq T(x)\},$$

hence by (i), (iii), and (ii), r is a measurable function, and similarly so is s. Also, $\{x: m(x, p(x)) > b\} = \{x: r(x) < p(x) < s(x)\}$, so $m(\cdot, p(\cdot))$ is measurable.

For $\delta \geqq 0$, $p + \delta$ is a measurable function and by what we have just proved, so is $m(\cdot, p(\cdot) + \delta)$; hence, $m^+(\cdot, p(\cdot))$ is measurable and similarly so is $m^-(\cdot, p(\cdot))$. Also,

$$\{(x, y): p(x) \leqq y \leqq \psi(x, p(x))\}$$
$$= \{(x, y): [p(x) = y \text{ and not } m^-(x, p(x)) = m^+(x, p(x))] \text{ or}$$

$$[p(x) \leqq y \in Y(x),\ m(x,y) = m(x,p(x)) + [y - p(x)]m^+(x,p(x)),$$
$$\text{and } m^-(x,p(x)) = m^+(x,p(x))]\},$$

which is $\mu \times \mathscr{L}$ measurable by (i) and the proven part of (iv). Thus, $\psi(\cdot, p(\cdot))$ is measurable by (i) of Lemma 3, (i), and (ii), which completes the proof.

THEOREM 5. *Suppose* $q^* \in \Phi$, $\lambda \in \mathscr{E}_1$, $|E(q^*)| < \infty$, $|C(q^*)| < \infty$, *and* μ *has finite substance* (see [6, § 2]). *Then for* (q^*, λ) *to satisfy (strongly satisfy) the functional multiplier rule, it is necessary and sufficient that* (q^*, λ) *satisfy (strongly satisfy) the pointwise multiplier rule.*

Proof. Sufficiency follows from Theorem 2.1 of [6].

Suppose (q^*, λ) satisfies the functional multiplier rule. Let

$$S = \{x : m^+(x, q^*(x)) > \lambda\} \quad \text{and} \quad R = \{x : m^-(x, q^*(x)) < \lambda\}.$$

By Lemma 4 (iv), S and R are measurable. We shall show $\mu(S) = \mu(R) = 0$.

Supposing $\mu(S) > 0$, choose a measurable $P \subset S$ such that $0 < \mu(P) < \infty$ and suppose $x \in P$ (whence $q^*(x) < U(x)$ since $m^+(x, q^*(x))$ exists). Let

$$p(x) = \inf\{y : m^+(x,y) < \lambda \text{ or } [U(x) = \infty \text{ and } y = \psi(x, q^*(x) + 1)]$$
$$\text{or } [U(x) < \infty \text{ and } y = \psi(x, q^*(x) + \tfrac{1}{2}[U(x) - q^*(x)])]\}.$$

By Lemma 3(i), $q^*(x) < p(x) \in Y(x)$. If $p(x) = \inf\{y : m^+(x,y) < \lambda\}$, it is easily shown that $p(x)$ is an extremizing point of $m(x, \cdot)$, and otherwise this holds by Lemma 3(ii). Thus, $m(x, p(x)) = e(x, p(x))$ by Assumption 1(iii). Also, $m^+(x,y) \geqq \lambda$ for $q^*(x) \leqq y < p(x)$, and there exists z such that $q^*(x) < z < p(x)$ and $m^+(x,y) > \lambda$ for $q^*(x) \leqq y \leqq z$.

By Lemma 4(i), (iii), and (iv), $\{(x,y) : q^*(x) \leqq y \leqq p(x)\}$ is $\mu \times \mathscr{L}$ measurable. Hence by Lemma 4(i), (ii), and (iv), p and $m(\cdot, p(\cdot))$ are measurable. Thus for $x \in P$, since $m(x, \cdot)$ is absolutely continuous on $[q^*(x), p(x)]$ by § 6.3 of [4],

$$l_\lambda(x, p(x)) = m(x, p(x)) - \lambda p(x)$$

(1) $$= \int_{q^*(x)}^{p(x)} [m^+(x,y) - \lambda]\, dy + m(x, q^*(x)) - \lambda q^*(x)$$

$$> m(x, q^*(x)) - \lambda q^*(x) \geqq l_\lambda(x, q^*(x)).$$

Define $q(x) = p(x)$ for $x \in P$ and $q(x) = q^*(x)$ for $x \in X - P$. Then $q \in \Xi$ and by (1) and Theorem 2.2((i) implies (ii)) of [6], (q^*, λ) does not satisfy the functional multiplier rule, in contradiction. Thus $\mu(S) = 0$. Similarly (symmetrizing the definition of ψ and the statement of Lemmas 3 and 4), $\mu(R) = 0$. Thus

(2) $$\text{not } [m^+(x, q^*(x)) > \lambda \text{ or } \lambda > m^-(x, q^*(x))] \quad \text{for } x \in X.$$

Let $Q = \{x : m(x, q^*(x)) > e(x, q^*(x))\}$. Then Q is measurable. For $x \in Q$, $q^*(x)$ is not an extremizing point of $m(x, \cdot)$, so by (2), $m^+(x, q^*(x)) = m^-(x, q^*(x)) = \lambda$. Define $q_0(x) = \psi(x, q^*(x))$ for $x \in Q$ and $q_0(x) = q^*(x)$ for $x \in X - Q$. Then for $x \in Q$, $q_0(x) \in Y(x)$ and $m(x, q_0(x)) = e(x, q_0(x))$ by Lemma 3, so

$$\begin{aligned} l_\lambda(x, q_0(x)) &= m(x, q_0(x)) - \lambda q_0(x) \\ &= m(x, q^*(x)) + [q_0(x) - q^*(x)]m^+(x, q^*(x)) - \lambda q_0(x) \\ &= m(x, q^*(x)) - \lambda q^*(x) > e(x, q^*(x)) - \lambda q^*(x) = l_\lambda(x, q^*(x)). \end{aligned}$$

Also, $q_0 \in \Xi$ by Lemma 4(iv). Hence by Theorem 2.2((i) implies (ii)) of [6], $\mu(Q) = 0$, so $m(\cdot, q^*(\cdot)) = e(\cdot, q^*(\cdot))$. From this, (2), and Lemma 2((v) implies (i)), we have $l_\lambda(x, q^*(x)) \geqq l_\lambda(x, y)$ for $y \in Y(x)$, $x \in X$, proving the theorem.

COROLLARY 6. *If the hypothesis of Theorem 5 holds, μ is nonatomic, and $C(T) < C(q^*) < C(U)$, then for q^* to be optimal (strongly optimal), it is necessary and sufficient that for some $\lambda \in \mathscr{E}_1$, (q^*, λ) satisfy (strongly satisfy) the pointwise multiplier rule.*

Proof. This follows from Corollary 3.3 of [6] and Theorem 5.

Remark 7. We show by examples that Theorem 5 and Corollary 6 cannot be strengthened in certain ways. Referring to the example of Remark 5.5 of [6], let

$$m(x, y) = 1 \quad \text{if } |y| \leqq 1 \quad \text{and} \quad m(x, y) = 2 - |y| \quad \text{if } 1 \leqq |y| \leqq 2 \qquad \text{for } (x, y) \in \Omega;$$

then $e(x, \cdot)$ has the concave envelope $m(x, \cdot)$ (whose graph is an isoceles trapezoid) for $x \in X$, and m is continuous. However, e is not coverable, since $m(x, 1) = m(x - 1) = 1 \neq 0 = e(x, 1) = e(x, -1)$ for $x \in X$, so (iii) of Assumption 1 fails. All other hypotheses of Theorem 5 are satisfied, but as noted in [6], the necessity conclusions fail. Therefore, in Theorem 5 one may not replace (iii) of Assumption 1 by the condition that e is measurable and m is continuous. By redefining $e(x, -1) = e(x, 1) = 1$ for $x \in X$, e becomes coverable, and Theorem 5 applies. Incidentally, if $r(x) = \frac{1}{2}$ for $x \in X$, then $e(\cdot, r(\cdot))$ is not measurable, but $m(\cdot, r(\cdot))$ and r are integrable.

To see that Theorem 5 fails if (iv) is omitted from Assumption 1, in Remark 5.5 of [6] redefine Ω to be $[0, 1] \times \mathscr{E}_1$ and $\mathscr{A}$ to be

$$\{(x, y) : [x \in D \text{ and } |y| = 2i] \text{ or } [x \in X - D \text{ and } |y| = 2i + 1] \text{ for some } i \in \omega\} \\ - X \times \{0\},$$

let $e(x, 0) = \frac{1}{2}$ for $x \in X$, elsewhere on Ω let e be the indicator function of $\mathscr{A}$, and proceed as before.

In Theorem 5 we assume $\mu \times \mathscr{L}$ measurability of m but not of e. To see that the latter would not insure the former, redefine $\mathscr{A} = D \times \{0\}$ and let e be the indicator function of $\mathscr{A}$.

THEOREM 8. *Suppose* (i) *$Y(x)$ is compact for $x \in X$,* (ii) *μ is nonatomic,* (iii) *$T \in \Phi$, $U \in \Phi$,* (iv) *$-\infty < C(T) \leqq C(U) < \infty$, and* (v) *$C(T) \leqq v \leqq C(U)$. Then there exist an optimal $q^* \in \Phi$ such that $C(q^*) = v$ and $p^* \in \Phi$ such that $C(p^*) \leqq v$ and $E(p^*) = \max\{E(p) : C(p) \leqq v\}$.*

Proof. For $x \in X$ and $\lambda \in \mathscr{E}_1$, define

$$\varphi_u(x, \lambda) = \sup\{y : y = T(x) \text{ or } m^+(x, y) \geqq \lambda\},$$

$$\varphi_l(x, \lambda) = \inf\{y : y = U(x) \text{ or } m^+(x, y) \leqq \lambda\}.$$

Then by (i),

(3) $$-\infty < T(x) \leqq \varphi_l(x, \lambda) \leqq \varphi_u(x, \lambda) \leqq U(x) < \infty \quad \text{for } x \in X, \quad \lambda \in \mathscr{E}_1.$$

Suppose $\lambda \in \mathscr{E}_1$. By (iv), μ is σ-finite over $\{x : U(x) > T(x)\}$. We have

$$\{(x, y) : y \leqq \varphi_u(x, \lambda)\} = \{(x, y) : y \leqq T(x) \text{ or } m^+(x, y) \geqq \lambda\},$$

which is $\mu \times \mathscr{L}$ measurable by Lemma 4(iii) and (i), so by Lemma 4(ii), $\varphi_u(\cdot, \lambda)$ is measurable and, by (iii), (iv), and (3), integrable; similarly, so is $\varphi_l(\cdot, \lambda)$. Define

$$I_u(\lambda) = \int_X \varphi_u(x, \lambda)\, d\mu x \quad \text{and} \quad I_l(\lambda) = \int_X \varphi_l(x, \lambda)\, d\mu x.$$

For $x \in X$, $\varphi_l(x, \cdot)$ is right continuous and $\varphi_u(x, \cdot)$ is left continuous. Thus, I_l is right continuous and I_u is left continuous by the monotone convergence theorem. By (3),

(4) $$C(T) \leqq I_l(\lambda) \leqq I_u(\lambda) \leqq C(U).$$

Since $\lim_{\lambda \to \infty} \varphi_u(x, \lambda) = T(x)$ for $x \in X$, by the dominated convergence theorem and (4) we have $\lim_{\lambda \to \infty} I_u(\lambda) = C(T)$, hence $\lim_{\lambda \to \infty} I_l(\lambda) = C(T)$. Similarly, $\lim_{\lambda \to -\infty} I_l(\lambda) = \lim_{\lambda \to -\infty} I_u(\lambda) = C(U)$.

Obviously, if $v = C(T)$ or $v = C(U)$, then T or U would respectively serve for q^*. Therefore, we assume that $C(T) < v < C(U)$. There exists a λ_0 such that

$$I_u(\lambda_0) = \lim_{\lambda \uparrow \lambda_0} I_u(\lambda) \geqq v \geqq \lim_{\lambda \downarrow \lambda_0} I_u(\lambda).$$

By the right continuity of I_l, $I_l(\lambda_0) \leqq v$ so for some $0 \leqq \alpha \leqq 1$, $v = \alpha I_u(\lambda_0) + (1 - \alpha) I_l(\lambda_0)$.

By Theorem 3.1 and Remark 3.5 of [6], (iv), and (3), we obtain a measurable $P \subset X$ and an integrable q^* such that $q^*(x) = \varphi_u(x, \lambda_0)$ for $x \in P$, $q^*(x) = \varphi_l(x, \lambda_0)$ for $x \in X - P$, and

$$C(q^*(x)) = \alpha C(\varphi_u(\cdot, \lambda_0)) + (1 - \alpha) C(\varphi_l(\cdot, \lambda_0)) = \alpha I_u(\lambda_0) + (1 - \alpha) I_l(\lambda_0) = v.$$

For $x \in X$, by the definitions of φ_l and φ_u, $q^*(x)$ is an extremizing point of $m(x, \cdot)$; thus by Assumption 1 (iii) and the same definitions,

(5) $$e(x, q^*(x)) = m(x, q^*(x)) \text{ and not } [m^+(x, q^*(x)) > \lambda_0 \text{ or } m^-(x, q^*(x)) < \lambda_0].$$

By Lemma 4 (iv), $m(\cdot, q^*(\cdot))$, i.e., $e(\cdot, q^*(\cdot))$, is a measurable function. We may assume there exists $\hat{q} \in \Phi$ such that $E(\hat{q}) > -\infty$. By virtue of (iv), $l_{\lambda_0}(\cdot, \hat{q}(\cdot))$ is integrable and $L_{\lambda_0}(\hat{q}) > -\infty$. Thus, $l_{\lambda_0}(\cdot, q^*(\cdot))$ is integrable since it dominates $l_{\lambda_0}(\cdot, \hat{q}(\cdot))$. Hence by (iv), $e(\cdot, q^*(\cdot))$ is integrable, i.e., $q^* \in \Phi$. Since $-\infty < L_{\lambda_0}(\hat{q}) \leqq L_{\lambda_0}(q^*)$, $E(q^*) > -\infty$. Hence by (5), Lemma (2) ((v) implies (i)), and Theorem 2.1 of [6], q^* is optimal, as desired.

To obtain p^* define $s(x) = \inf\{y : \text{not } [m^-(x, y) < 0 \text{ or } 0 < m^+(x, y)]\}$ for $x \in X$. Then $m(x, s(x)) \geqq m(x, y)$ for $y \in Y(x)$, $x \in X$, and $m(\cdot, s(\cdot)) = e(\cdot, s(\cdot))$ by (iii) of Assumption 1. It is easily shown by Lemma 4 (iii) and (ii) that s is measurable, whence by Lemma 4 (iv), so is $m(\cdot, s(\cdot))$. Since $m(\cdot, s(\cdot)) \geqq e(\cdot, \hat{q}(\cdot))$ and $E(\hat{q}) > -\infty$, $m(\cdot, s(\cdot))$ is integrable. Thus, $E(s) = \int_X m(x, s(x))\, d\mu x \geqq E(q)$ for $q \in \Phi$.

Now let $a = \sup\{E(q) : C(q) \leqq v\}$, $v(w) = \sup\{E(q) : C(q) = w\}$ for $w \in$ range C, and $F = \text{range } C \cap \{w : v(w) > -\infty\}$. By Theorem 3.1 and Remark 3.5 of [6], F and $\mathscr{E}_2 \cap \text{range } (C, E)$ are convex. We have shown that for $w \in F$, $w = C(q)$ and $v(w) = E(q)$ for some q. It follows from Theorem 6.10 of [6] that v is concave on interior F or ∞ on this set.

If $v \geqq \sup F$, s serves as p^*; $v \leqq \inf F$ is trivial. We may assume v is in interior F, on which ν is concave. Choose $p, q_1, q_2, \cdots \in \Phi$ such that $C(q_j) \leqq v$ for $j \in \omega$, $E(q_j) \to a$ and $C(q_j) \to C(p) \leqq v$. If $C(p) > \inf F$, then ν is continuous at $C(p)$, so $a = \nu(C(p))$ and, as we have shown, there exists p^* such that $E(p^*) = \nu(C(p)) = a$ and $C(p^*) = C(p)$. If $C(p) = \inf F$, since $a \geqq \nu(v)$ and ν is concave, we have $\nu^-(v) \leqq 0$ whence $C(s) \leqq v$. Again, s serves as p^*, which completes the proof.

Remark 9. In [5], Stone provides an existence result, Theorem 3.3, under conditions different from those of Theorem 8; he replaces the condition $C(U) < \infty$ (thereby permitting $U = \infty$) by the condition that $E(T)$ and $E(U)$ are finite and $e(x, \cdot)$ is nondecreasing for $x \in X$. Example 3.4 of [5] shows that neither $C(U) < \infty$ nor $C(T) > -\infty$ may be omitted from Theorem 8.

The following example demonstrates that in Theorem 8, Assumption 1 (iii) may not be replaced by the condition that e is a Borel function and $e(x, \cdot)$ is nondecreasing for $x \in X$. Let $X = [0, 1]$, and for $x \in X$, let $T(x) = 0$ and $U(x) = 2$. For $(x, y) \in \Omega$, let $e(x, y) = 0$ for $0 \leqq y \leqq 1$ and $e(x, y) = 1$ for $1 < y \leqq 2$. Note that if $C(q) = 1$, then $E(q) < 1$. However, $\sup\{E(q): C(q) = 1\} = 1$.

Defining concave envelopes without requiring continuity at endpoints would invalidate Theorem 8. To see this, let $\Omega = [0, 1] \times [0, 1]$, and for $(x, y) \in \Omega$, let $e(x, y) = y^2$ and $m(x, y) = y$ for $y \in (0, 1]$ and $e(x, 0) = m(x, 0) = -1$. Then m would cover e since the only extremizing points of $m(x, \cdot)$ are 0 and 1, for $x \in [0, 1]$. However, there is no optimal $q^* \in \Phi$ such that $C(q^*) = \frac{1}{2}$.

Remark 10. Our use of concave envelopes was originally motivated in part by Arkin [1]. In [1], $e(x, \cdot)$ is assumed to be a probability distribution function. However, we point out that the proof of necessity in [1] for the case where $e(x, \cdot)$ is not concave is incomplete in that it is shown merely that (our notation) if q^* is optimal with respect to E and C and (defining $M(q) = \int_X m(x, q(x))\, d\mu x$ when this exists) *if* there exists a function r^* which is optimal with respect to M and C with $C(r^*) = C(q^*)$, then there is an r^{**} such that $C(r^{**}) = C(r^*) = C(q^*)$ and $M(r^*) = M(r^{**}) = E(r^{**}) = E(q^*)$, and hence that r^{**} is optimal with respect to E and C. It remains to show that such r^* exists and that $q^*(x)$ maximizes a Lagrangian for $x \in X$.

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