SEARCH THEORY: A MATHEMATICAL THEORY FOR FINDING LOST OBJECTS

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Reprinted from the MATHEMATICS MAGAZINE
Vol. 50, No. 5, November 1977
pp. 248–256

Search Theory: A Mathematical Theory for Finding Lost Objects

A model for determining an optimal search pattern for a wind-driven ship lost at an uncertain position.

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A ship is in distress and you are a Coast Guard search and rescue coordinator whose job is to help save the ship. A last faint radio message from the ship says that due to electrical problems, its navigation equipment has failed and has allowed the ship to wander off course and strike a submerged rock. The ship is taking on water faster than it can be pumped out, and the ship's captain estimates that the boat can stay afloat for only one or two days. He requests extra pumps and a tow to port in order to save the ship. The captain cannot say exactly where he is, but he knows the general region in which he is located.

You have one aircraft available to look for the ship and drop the needed pumps. Considering the number of hours of daylight, the endurance of the aircraft, and the time required to travel to and from the vicinity of the lost ship, you find that there are 10 hours of usable search effort available per day for the next two days. How should these 20 hours of search be spent in order to maximize the probability of finding the ship before it sinks? Which areas should be searched first and for how long? What is the total amount of time that should be spent in each area? These questions can be answered by the use of search theory; since you are a search and rescue coordinator, you can use your remote terminal to interrogate the Coast Guard's central computer to find the answers. Let's see how search theory, via the Coast Guard computer, provides the answers.

First, we must define the problem more carefully by specifying a probability distribution for the target's location and a detection function which relates hours of searching to probability of detection.

Target location distribution

Suppose, for simplicity, that the general region in which the ship is located is rectangular, and subdivided into six rectangular subregions or cells each with area A, as shown in FIGURE 1. Based on your knowledge of the ship's intended route and the captain's description of his location, you feel that cell 1 is distinctly the most likely to contain the ship and the other cells are uniformly less likely to contain it. On this basis, you assign a subjective probability $p(1)=.5$ to the target being in cell 1 and $p(j)=.1$ to the target being in cell j for $j\neq 1$. Within each cell, you assume the ship's location distribution is uniform.

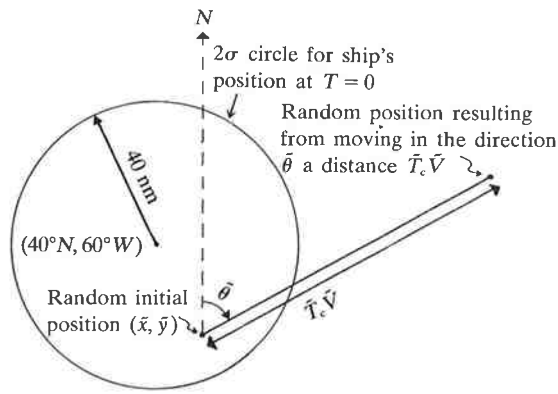
Devising a probability distribution for a target's location is an art rather than a science. The procedure relies on the subjective judgment and experience of the search planner aided by the facts available, such as the last known position, the accuracy of the navigation system employed, and the intended route of the missing object. Indeed, construction of the target location distribution is often the most crucial step in preparing a search plan. For this reason, the Coast Guard central computer contains a number of programs which assist the search planner in constructing these distributions. These programs are designed to account for the uncertainty in the location information and to update past information to the present time by allowing for drift caused by wind and ocean currents.

As an example of a more realistic prior target location distribution, let us suppose that a ship was last reported at 40°N, 60°W heading east at 10 knots. The ship was expected to report its position once a day, but for two days nothing was heard from it. Let $T=0$ correspond to the time of the last reported position. We shall assume a casualty has occurred some time between $T=0$ and $T=24$ hours, since otherwise we would have expected to hear from the ship at $T=24$ hours. A probability distribution for the target's location could be constructed as follows. If the navigational equipment used by the ship has a circular normal error with standard deviation 20 nm in any direction, then we can use that distribution for the target's location at $T=0$. That is, let us take the reported position to be the origin of a two-dimensional plane which locally represents the surface of the earth. Let the y-axis be oriented north-south and the x-axis east-west. Then the x-coordinate of the actual position at $T=0$ is normally distributed with mean 0 and standard deviation 20 nm. Similarly, the y-coordinate has an independent normal distribution with mean 0 and standard deviation 20 nm.

Cell 1 $p(1)=.5$	Cell 2 $p(2)=.1$	Cell 3 $p(3)=.1$
Cell 4 $p(4)=.1$	Cell 5 $p(5)=.1$	Cell 6 $p(6)=.1$

Search region and cells

FIGURE 1



A sample replication

FIGURE 2

Although the reported heading is due east at 10 knots, we assume that the ship's actual heading is uniformly distributed between 80° and 100° and that its speed is uniformly distributed between 8 and 12 knots. Finally, since we do not know the time T_c at which the casualty occurred, we assume that T_c is uniformly distributed between 0 and 24 hours.

A probability map for the ship's position at the time of the casualty can be generated by a Monte Carlo technique as follows. For each replication, random draws are made to determine the ship's initial position $(\tilde{x}, \tilde{y})$, the ship's heading $\tilde{\theta}$, speed $\tilde{V}$, and the length of time $\tilde{T}_c$ to casualty. The ship's position is then advanced from its initial position $(\tilde{x}, \tilde{y})$ in the direction $\tilde{\theta}$ a distance $\tilde{T}_c\tilde{V}$ as shown in FIGURE 2. The resulting position represents one replication. This process is repeated 10,000 times, to obtain the probability distribution shown in FIGURE 3. The unknown length of time till the distress occurred results in a distribution which is elongated in the east-west direction compared to the circular normal distribution for the target's last known position. In the Coast Guard programs, these points are further moved to account for the action of winds and currents on a vessel adrift.

Although we could pursue the above example to show how optimal allocations of effort are obtained, for the purpose of presenting the essentials of search therory we shall return to the simple distribution in FIGURE 1. To make use of this (or any) distribution for the target's location, we must determine a function that relates effort spent to probability of detection. Such a function is called, naturally enough, a **detection function**; we turn now to a study of its properties.

	W 62D OM – W 61D OM	W 61D OM – W 60D OM	W 60D OM – W 59D OM	W 59D OM – W 58D OM	W 58D OM – W 57D OM	W 57D OM – W 56D OM	W 56D OM – W 55D OM	W 55D OM – W 54D OM	W 54D OM – W 53D OM	W 53D OM – W 52D OM
N42D OM – N41D OM			2	5	6	15	29	15	5	
N41D OM – N40D OM	5	176	804	944	972	905	749	309	39	1
N40D OM – N39D OM	1	160	789	994	958	978	737	282	47	
N39D OM – N38D OM			2	7	6	12	20	23	3	

Probability map of distress locations (Probabilities are multiplied by 10,000.)

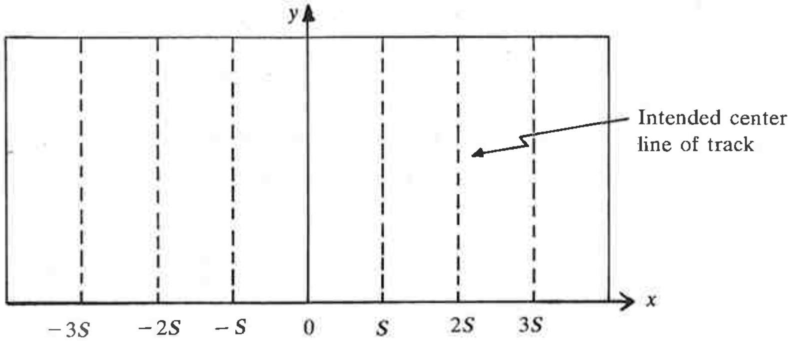
FIGURE 3

The detection function

To obtain a detection function for this example, we assume that there is a detection range d such that the probability of detection is β if the aircraft passes within distance d of the ship, and 0 otherwise. Let $W = 2\beta d$ be the sweep width of the search, and $M = 2d$ be the width of the search path. The sweep width W is a measure of the sensor's average detection capability in the sense that if the search vehicle were to travel in a straight line for a distance l through a region of area A in which there was a uniform distribution of targets all having the same detection characteristics as the ship, then Wl/A fraction of these targets would be detected.

When the aircraft searches one of the cells or rectangles, the pilot will attempt to follow a parallel path search with intended spacing S between tracks as shown in FIGURE 4. Suppose that $\beta = 1$, i.e., the probability of detection is 1 if the aircraft passes within distance d of the ship, that $S = M$, and that the aircraft can follow the intended tracks exactly. In this case the probability of detection as a function of search time t, assuming the target is in the rectangle, would follow the straight line in FIGURE 5.

However, two factors intervene to make the actual detection probability fall below this line. First, experience shows that because of fatigue, boredom, and distractions, an observer can pass within visual sighting range of a target and still not see it. Thus, normally, we are dealing with a situation in which $\beta < 1$. Second, there are usually substantial navigational errors involved in the placement of the tracks, especially when flying over the ocean. It is not uncommon for the search aircraft to have navigational uncertainties with a standard deviation as large as 10 miles (see section 612 of [**8**]). The result of these navigation or track placement errors is to cause overlaps in the coverage of some paths and gaps in others. Thus, even when $\beta = 1$, the probability of detection falls below the straight line in FIGURE 5.



Nominal track placement for parallel path search

FIGURE 4

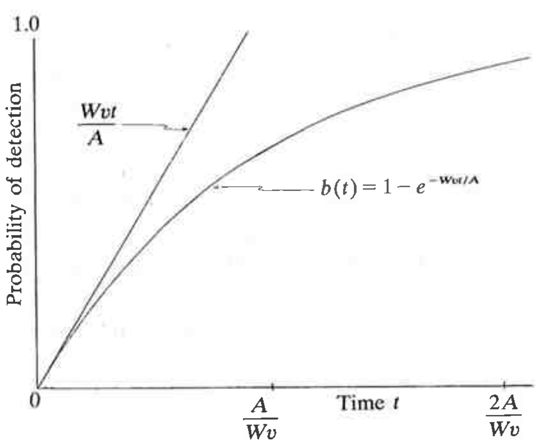
In order to obtain detection probabilities for a parallel path search, we use the following simple model developed by R. K. Reber [**4**]. Suppose that the search tracks are indeed parallel to the y-axis as shown in FIGURE 4 and that iS is the intended value of the x-coordinate of the ith path. However, the actual value $\bar{x}_i$ is normally distributed with mean iS and standard deviation σ.

Let $g_i(x)$ be the probability that the point (x, y) in the rectangle is covered by the ith path. Then

$$
\begin{aligned}
g_i(x) &= Pr\{\bar{x}_i - d \leqq x \leqq \bar{x}_i + d\} = Pr\{x - d \leqq \bar{x}_i \leqq x + d\} \\
&= \frac{1}{\sigma\sqrt{2\pi}}\int_{x-d}^{x+d} \exp[-(u-iS)^2/2\sigma^2]du = \frac{1}{\sqrt{2\pi}}\int_{(x-iS-d)/\sigma}^{(x-iS+d)/\sigma} e^{-\frac{1}{2}z^2}dz.
\end{aligned}
\tag{1}
$$

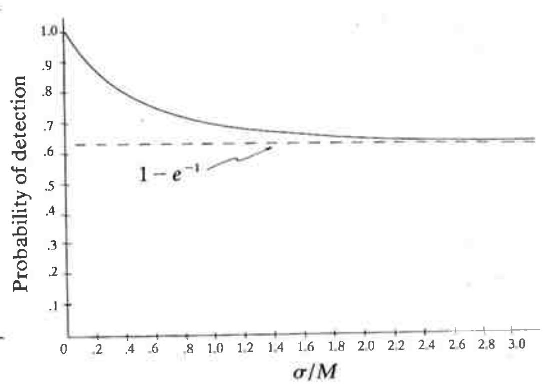
Since the target will be detected with probability β if it falls within the sensor path, $\beta g_i(x)$ is the probability that the target will be detected by the ith track given it is located at (x, y).

Assuming that each path presents an independent chance at detecting the target, the probability of detecting the target on at least one of the $2N+1$ paths covering the rectangle given it is located at (x, y) in the rectangle is $h(x) = 1 - \Pi_{i=-N}^{N}[1 - \beta g_i(x)]$. For the remainder of this calculation, we shall assume that x is sufficiently far from the boundaries of the rectangle that we can replace the finite



Detection probability for random search

FIGURE 5



Detection probability for parallel path search

FIGURE 6

product in the above equation by the following infinite one so that

$$h(x)=1-\prod_{i=-\infty}^{\infty}[1-\beta g_i(x)]. \tag{2}$$

Ignoring edge effects, one can see from (1) and (2) that the function h is periodic with period S equal to the spacing of the tracks. Thus, the average probability of detection over the rectangle (ignoring edge effects) is

$$Q\left(\frac{M}{\sigma},\beta,\frac{S}{\sigma}\right)=\frac{1}{S}\int_{-S/2}^{S/2}h(x)dx=\frac{\sigma}{S}\int_{-S/2\sigma}^{S/2\sigma}h(\sigma y)dy. \tag{3}$$

Observe that by the way we have written equation (3), we are claiming that Q is a function of only M/σ, β, and S/σ. By referring back to equations (1) and (2), the reader may check that this is true.

The probability Q is plotted as a function of $(M/\sigma)^{-1}$ in FIGURE 6 for the case where $\beta=1$ and $M=S$. Notice that as M/σ becomes small, Q approaches $1-e^{-1}\approx 0.63$. In fact, it can be shown that for arbitrary M, β, and S,

$$\lim_{\substack{M/\sigma\to 0\\ S/\sigma\to 0}} Q\left(\frac{M}{\sigma},\beta,\frac{S}{\sigma}\right)=1-e^{-\beta M/S}, \tag{4}$$

provided M/S remains constant. From FIGURE 6 we can see that this limiting case is approached quite quickly when $\beta=1$ and $M=S$. Numerical experience indicates that when $\beta M/\sigma$ and $S/\sigma\leqq\frac{1}{2}$, equation (4) gives a good and somewhat conservative approximation to Q. Since we have already noted that one can have navigational errors with standard deviations of 10 miles or greater and since in our example we will be dealing with visual search with $\beta M=3.2$ miles, we shall assume from here on that the limit in (4) holds. In practice, one usually chooses $S\leqq M$ so that there are no intentional gaps in the coverage.

For our purposes it is most convenient to restate (4) in terms of the time t spent in the search area. Suppose the search rectangle has length a_1 and width a_2. Then $A=a_1a_2$ is the area of the rectangle and $A/S=a_1a_2/S$ is the track length required for parallel path search of the rectangle with tracks spaced a distance S apart. Suppose that the aircraft travels at an average speed of v; then $t=A/Sv$ is the time spent performing such a search. Recalling that $W=\beta M$, we have $\beta M/S=W/S=Wvt/A$ and $b(t)$, the probability of detection by time t, becomes

$$b(t)=1-\exp(-Wvt/A)\quad\text{for } t\geqq 0. \tag{5}$$

This is the random search formula of B. O. Koopman which was originally obtained in **[3]** by a heuristic but well motivated argument. Experience has shown that the exponential detection function defined in (5) provides reasonable and conservative estimates of detection capability in many search situations. This fact and its simple form account for its widespread use in search problems. For systematic searches of an area, reality is bounded by the two curves in FIGURE 5.

An important property of the random search formula is shown in FIGURE 5. Initially, the slope of b is Wv/A since at the beginning of the search there is almost no chance of overlapping with previous search. However, as the search continues, the chance of overlap with previously searched areas of the rectangle increases, and this slows the rate at which the probability of detection increases. Consequently, the random search formula exhibits a diminishing rate of return. This diminishing rate (or decreasing derivative) of b will play an important role in finding the optimal allocation in the next section.

Optimal allocation of effort

We can now find the optimal allocation of effort for rescuing the sinking ship. Suppose that the boat is 50 feet long, that each cell is a 50–mile by 50–mile square and that the meteorological visibility is three miles over the entire region. (Meteorological visibility is the maximum distance at which very large objects such as mountains can be seen.) Checking Figure 7–2 of [8] we find that if the aircraft flies at the optimum height for this situation, it will have a visual sweep width of $W = 3.2$ nm. The aircraft searches at a speed of 100 knots, so $A = 2500$ (nm)^2, $v = 100$ knots, and $W = 3.2$ nm.

If the aircraft spends time t_i searching in cell i for $i = 1, \ldots, 6$, then the probability of detecting the ship with this allocation of time is

$$\sum_{j=1}^{6} p(j)b(t_j) = \sum_{j=1}^{6} p(j)(1 - e^{-Wvt_j/A}).$$

We shall assume that the time required to switch from one cell to the other is negligible, so that the total time (or cost) required by this allocation is $\Sigma_{j=1}^{6} t_j$. In search problems, the time required to complete a search is often taken to be the cost of the search.

More generally, let f be an allocation of time, i.e., $f(j) \geqq 0$ is the amount of time spent looking in cell j for $j = 1, \ldots, 6$. Define

$$P[f] = \sum_{j=1}^{6} p(j)b(f(j)), \tag{6}$$

$$C[f] = \sum_{j=1}^{6} f(j). \tag{7}$$

Then $P[f]$ gives the probability of detection and $C[f]$ gives the cost (or total time) associated with the allocation f. Let us consider the optimal allocation of K hours of search. That is, we attempt to find an allocation f^* such that $C[f^*] \leqq K$ and

$$P[f^*] = \text{maximum of } P[f] \text{ over all allocations } f \text{ such that } C[f] \leqq K. \tag{8}$$

Such an f^* is called optimal for cost K.

Naively, one might suppose that, since the cells are of equal size and the sweep width is equal in all cells, one should allocate his effort in proportion to the probability $p(j)$ of the target being in cell j for $j = 1, \ldots, 6$. However, as we shall see below, this does not produce an optimal allocation. The reason for this lies in the decreasing rate of return exhibited by the detection function b.

Suppose the aircraft has spent time t_j looking in cell j for $j = 1, \ldots, 6$ and is considering spending a small increment of time h looking in cell j. The increase in probability of detection resulting from this increment is approximately $p(j)b'(t_j)h$. In the short run, the searcher would benefit most from placing the increment in the cell having the highest value of $p(j)b'(t_j)$.

Define $\rho_j(t) = p(j)b'(t)$ for $j = 1, \ldots, 6$ and $t \geqq 0$. Then ρ_j is called the rate of return function for cell j. Because the function b exhibits a decreasing rate of return, the rate ρ_j is similarly decreasing. Let us examine the search policy which always places the next small increment of effort in the cell having the highest rate of return. This rate will depend on t_j, the time previously spent looking in cell j for $j = 1, \ldots, 6$. Under this policy, the next increment goes into the cell j^* that satisfies $\rho_{j^*}(t_{j^*}) = \max\{\rho_j(t_j) \mid j = 1, 2, \ldots, 6\}$. Such a search policy is called **locally optimal.** We shall show below that following a locally optimal policy for t hours yields an optimal allocation of t hours of effort for all t; we will do this in particular for $t = 10$ and $t = 20$ hours.

At time 0 when no search has been made in any cell, $\rho_j(0) = p(j)b'(0) = p(j)Wv/A$. Since $p(1) = .5$ and $p(j) = .1$ for $j \neq 1$, the locally optimal policy calls for looking solely in cell 1 until the time s that satisfies

$$\rho_1(s) = .5\,\frac{Wv}{A}\,e^{-Wvs/A} = .1\,\frac{Wv}{A} = \rho_j(0) \text{ for } j \neq 1,$$

i.e.,

$$s = \ln 5\,\frac{A}{Wv} = 7.81 \ln 5 \text{ hrs} = 12.6 \text{ hrs}.$$

In order to continue the locally optimal policy beyond search time $s = 12.6$ hrs., the additional time h_j spent looking in cell j for $j = 1, \ldots, 6$, must be split so that $\rho_1(s + h_1) = \rho_j(h_j)$ for $j \neq 1$. In other words,

$$\rho_1(s + h_1) = .5\,\frac{Wv}{A}\,e^{-Wv(s+h_1)/A} = .1\,\frac{Wv}{A}\,e^{-Wvh_1/A} = .1\,\frac{Wv}{A}\,e^{-Wvh_j/A} = \rho_j(h_j),$$

which implies $h_j = h_1$, for $j = 1, \ldots, 6$.

This search policy may be described as follows: Let $\varphi^*(j, t)$ be the number of hours out of the first t that are spent looking in cell j for $j = 1, \ldots, 6$. Then

$$\varphi^*(1, t) = \begin{cases} t & \text{for } 0 \leqq t \leqq 12.6 \\ 12.6 + \frac{1}{6}(t - 12.6) & \text{for } 12.6 < t < \infty \end{cases}$$

(9)
$$\varphi^*(j, t) = \begin{cases} 0 & \text{for } 0 \leqq t \leqq 12.6 \\ \frac{1}{6}(t - 12.6) & \text{for } 12.6 < t < \infty,\ j = 2, \ldots, 6. \end{cases}$$

Following the locally optimal plan φ^* would require splitting effort instantaneously and uniformly among the six cells if the search progressed past 12.6 hours. Obviously the aircraft cannot do this. However, if the search planner has t hours of search effort available, he can calculate $\varphi^*(j, t)$, the total search time that accumulates in cell j during the time interval t. He could then allocate the aircraft's search effort so that it spends $\varphi^*(j, t)$ hours searching in cell j for $j = 1, \ldots, 6$. We now show that if the planner does this for $t = 10$ and $t = 20$ hours, he will obtain an optimal allocation for 10 and 20 hours of search respectively.

To show that one obtains an optimal allocation for 10 hours, let $f^*(1) = \varphi^*(1, 10) = 10$ and $f^*(j) = \varphi^*(j, 10) = 0$ for $j = 2, \ldots, 6$, so that f^* gives the allocation resulting from following the plan φ^* for 10 hours. While the plan φ^* has the virtue of always adding search effort to maximize the short-term gain, it is not obvious that such a policy produces an optimal allocation of the total effort available. It is conceivable that a plan that considered the total 10 hours of search available for the first day might produce a higher probability of detection at the end of this 10 hours than that produced by following φ^*. However, by using Lagrange multipliers we can show that f^* is optimal for cost 10 hours.

Consider the function l defined by

(10)
$$l(j, \lambda, t) = p(j)\,b(t) - \lambda t \text{ for } j = 1, \ldots, 6,\ \lambda \geqq 0, \text{ and } t \geqq 0.$$

This function is called the pointwise Lagrangian, and λ is called a Lagrange multiplier. Let $\lambda = \rho_1(f^*(1))$. Then by the nature of the locally optimal plan, $\lambda \geqq \rho_j(f^*(j)) = \rho_j(0)$ for $j = 2, \ldots, 6$.

We shall now show that

(11)
$$l(j, \lambda, f^*(j)) = \text{maximum of } l(j, \lambda, t) \text{ over } t \geqq 0 \text{ for } j = 1, \ldots, 6.$$

For $j = 1$, this is accomplished by taking the derivative l' of l with respect to t, recalling that b' is decreasing, and observing that

$$l'(1, \lambda, t) = \rho_1(t) - \lambda = p(1)b'(t) - \lambda \begin{cases} \geqq 0 \text{ for } 0 \leqq t \leqq f^*(1), \\ \leqq 0 \text{ for } f^*(1) < t < \infty. \end{cases}$$

For $j = 2, \ldots, 6$, we find that $l'(j, \lambda, t) = \rho_j(t) - \lambda \leqq 0$ for $t \geqq 0$, so that the maximum of $l(j, \lambda, \cdot)$ occurs at $t = 0$. This proves (11).

The fact that f^* satisfies (11) allows us to show that f^* is optimal for cost $K = 10$ hours as follows. Suppose that f is an allocation such that $C[f] = K = 10$ hours. Then by (11),

(12) $$l(j, \lambda, f^*(j)) \geqq l(j, \lambda, f(j)) \text{ for } j = 1, \ldots, 6.$$

Summing both sides of (12) on j and making use of (6), (7), and (10), one obtains $P[f^*] - \lambda C[f^*] \geqq P[f] - \lambda C[f]$, which implies $P[f^*] - P[f] \geqq \lambda (C[f^*] - C[f]) = 0$, where the last equality follows from $C[f^*] = C[f]$. Thus, f^* is optimal for cost $K = 10$ hours. One can check that $P[f^*] = .36$. (Observe that this optimization cannot be performed by using the standard Lagrange undetermined multiplier method, which involves solving for a critical point, because the optimum lies on the boundary of the domain of the function being maximized.)

We now know how to allocate the first day of search, but what if the search lasts beyond that? Returning to the plan φ^*, we see that by setting $\lambda = \rho_1(\varphi^*(1, 20))$, we can follow the same argument as above to show that $\varphi^*(1, 20) = 13.83$, and $\varphi^*(j, 20) = 1.23$ for $j = 2, \ldots, 6$, is optimal for cost $K = 20$ hours.

This is a fortunate result since it means that the optimal plan for 20 hours of search can be treated as a continuation of the optimal plan for 10 hours. The probability of detecting the target by the time 20 hours of search is completed is .49.

Uniformly optimal plans

A retrospective glance at the proof given above shows that for any $t \geqq 0$, $\varphi^*(\cdot, t)$ is optimal for cost t. We say, in this case, that φ^* is a **uniformly optimal** plan. Such a plan avoids the problem of sacrificing some probability of detection at intermediate times in order to obtain the maximum detection probability by 20 hours of search. However, as we noted above, it is not possible to follow the plan φ^* exactly. Instead, the coordinator will have to settle for some operationally reasonable approximation to the uniformly optimal plan if it is important to maintain high intermediate values of probability of detection. In the case of the sinking ship, it is indeed desirable to try to follow the uniformly optimal plan since the captain's figure of two days until the ship sinks is at best an estimate and could be optimistic.

Let us return to the plan that allocates effort in proportion to the prior probabilities $p(j)$ for $j = 1, 2, \ldots, 6$, and see how it compares with the optimal plan. The table in FIGURE 7 compares the probability of detection which results from following the proportional plan and the optimal plan.

Hours of Search	Optimal Plan	Proportional Plan	Percent Improvement of Optimal Plan over Proportional Plan
2	.11	.07	57
4	.20	.14	43
6	.27	.20	35
8	.32	.25	28
10	.36	.30	20
12	.39	.34	15
14	.42	.38	11
16	.44	.41	7
18	.47	.44	7
20	.49	.47	4

Probability of detection

FIGURE 7

Notice that the optimal plan yields the biggest improvement in the early hours of the search. This is important because early success increases the chance of rescuing a lost person alive, or of saving a ship before it goes down and decreases the cost of the search effort.

Usually the Coast Guard must deal with more complicated distributions than the one considered for the example. Often there are a large number of cells (e.g., 50) and it is commonplace for the sweep width to vary from cell to cell because of variations in visibility. Example 2.2.8 of [7] gives an algorithm for computing optimal allocations for an n-region target distribution and an exponential detection function which may vary from region to region. A similar algorithm is employed by the Coast Guard's computer-assisted search planning (CASP) system which is discussed further in [5].

Uniformly optimal search plans have been found for a wide variety of searches involving stationary targets. When the detection function has a continuous, positive and strictly decreasing derivative, the detection function is called regular. For regular detection functions, Theorems 2.2.4 and 2.2.5 of [7] give uniformly optimal search plans in a form readily adapted to computer calculation. For example, suppose there are J regions and that $p(j)$ is the probability that the target is in region j. Assume that the detection function b is regular. For $j = 1, \ldots, J$, let $\rho_j(t) = p(j)b'(t)$ for $t \geqq 0$, and let $\rho_j^{-1}(\lambda)$ be the inverse of ρ_j evaluated at λ when $0 < \lambda \leqq \rho_j(0)$ and 0 when $\rho_j(0) < \lambda$. Define $U(\lambda) = \Sigma_{j=1}^{J} \rho_j^{-1}(\lambda)$ for $\lambda > 0$ and $\phi^*(j, t) = \rho_j^{-1}(U^{-1}(t))$ for $t \geqq 0$, $j = 1, 2, \ldots, J$. Then ϕ^* is uniformly optimal among all plans ϕ which satisfy $\Sigma_{j=1}^{J} \phi(j, t) = t$ for $t \geqq 0$. The functions ρ_j and U and their inverses are easily calculated numerically. Occasionally, as in the case of a circular normal target distribution coupled with an exponential detection function, these functions can be computed analytically; see examples 2.2.1 and 2.2.7 of [7]. One might hope that uniformly optimal plans exist for a wide class of moving target problems in the same fashion as they do for stationary target problems. However, uniformly optimal plans are the exception rather than the rule for moving targets.

Search theory is a relatively new area of applied mathematics. It had its beginning during the years 1942–1945 in the work done by Bernard O. Koopman and his colleagues in the Anti-Submarine Warfare Operations Research Group of the U.S. Navy. A compendium of this work [3] was written in 1946 and still remains one of the basic references in the area. In fact, Koopman is in the process of revising and updating this work. In the years since 1946, the subject has grown and matured into a field with an extensive body of results. Yet the field still retains its motivation in terms of actual problems which suggest areas of research. For example, the 1968 search for the U.S. nuclear submarine Scorpion (see [6]) suggested work on search with uncertain sweep width and search in the presence of false targets. Work for the U.S. Coast Guard to develop computer-assisted search-planning programs suggested the study of conditionally deterministic motion, discussed in Chapter 8 of [7]. The interested reader can gain an introduction to a wide range of problems in search and detection theory by perusing [1] and [2], the published bibliographies on search and detection theory.

Much of the work in search theory has been supported by the Office of Naval Research, which has provided partial support for the writing of this paper under Contract No. N00014-76-C-0696.

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