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SEARCH FOR TARGETS WITH CONDITIONALLY DETERMINISTIC MOTION*

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Abstract. Optimal search for targets with conditionally deterministic motion is investigated. The target motion takes place in $\mathscr{Y}$, a copy of Euclidean n-space, and depends on a stochastic parameter ξ which takes values in $\mathscr{X}$, another copy of Euclidean n-space. The target motion is deterministic given knowledge of ξ. That is, there is a function $Y: T \times \mathscr{X} \to \mathscr{Y}$, where T is a time interval, such that $Y(\cdot, x)$ gives the target motion conditioned on $\xi = x$. Search plans are specified by functions $\mu: T \times \mathscr{Y} \to [0, \infty)$. A functional P is defined so that $P_t[\mu]$ gives the probability of detecting the target by time t using plan μ.

Let $J(t, x)$ be the absolute value of the Jacobian of $Y(t, \cdot)$ evaluated at x. If there exist functions $m: T \to (0, \infty)$ and $j: \mathscr{X} \to (0, \infty)$ such that $J(t, x) = m(t)j(x)$ for $(t, x) \in T \times \mathscr{X}$, the target motion is called *factorable*. Let $\varphi_2: T \to [0, \infty)$. If the target motion is factorable, Theorems 4.1 and 4.2 give a method for finding a plan μ^* such that $\int_{\mathscr{Y}} \mu^*(t, y)\,dy \leqq \varphi_2(t)$ for $t \in T$ and $P_t[\mu^*] \geqq P_t[\mu]$, $t \in T$, for all search plans μ satisfying $\int_{\mathscr{Y}} \mu(t, y)\,dy \leqq \varphi_2(t)$ for $t \in T$. Let k and l be positive numbers. Theorem 5.1 gives sufficient conditions for finding a plan μ^* such that $\mu^* \leqq k$, $\int_T \int_{\mathscr{Y}} \mu^*(t, y)\,dy\,dt \leqq l$ and $\lim_{t\to\infty} P_t[\mu^*] \geqq \lim_{t\to\infty} P_t[\mu]$ for any search plan μ satisfying $\mu \leqq k$ and $\int_T \int_{\mathscr{Y}} \mu(t, y)\,dy\,dt \leqq l$. Examples of optimal search plans are computed to illustrate the use of the above theorems.

1. Introduction. We consider search for a target whose motion is conditionally deterministic. That is, the target's motion depends on a stochastic parameter such as initial position or velocity. If this parameter were known, then the target's position would be known at all times in the future. Thus, the target's motion is deterministic, conditioned on knowledge of the parameter. Section 2 gives a mathematical description of the target's motion.

In the third section we define a class of problems which involve maximizing the probability of detecting the target while satisfying certain constraints on search effort. While we are not able to give a general method of solving this class of problems, we do single out two interesting subclasses and show how optimal plans can be found in these cases. The first class of problems involves factorable target motions. This notion of factorability is defined precisely in § 4, but the crucial requirement is that the Jacobian of the target motion factor into time-dependent and space-dependent parts. Theorem 4.1 gives the main result on finding optimal plans when the target motion is factorable. In the second class there is a constraint on the search density (in time and space) and on the total amount of search effort, but no constraint on the rate at which effort may be applied. For this class, Theorem 5.1 gives a sufficiency result which is useful for finding optimal plans. In both cases, examples are computed which illustrate the use of the theorems.

The example computed to illustrate the use of Theorem 5.1 yields the unexpected result that for the search problem considered there, the search area of the optimal plan first expands about the origin and then begins to contract back to the origin despite the fact that the target is continually moving away from the origin.

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The search problem considered in this paper differs from those moving target problems considered in [3], [4], [5] or [7]. In the above cases knowledge of the target's initial position (or some other parameter) does not specify, in a deterministic way, the target's subsequent motion. Our assumption of conditionally deterministic target motion, while more restrictive than the assumptions of the above references, allows us to obtain explicitly, in many cases, the optimal search plans in the classes considered in §§ 4 and 5.

2. Target motion. We consider a space $\mathscr{Y}$, in which the target moves, and a parameter space $\mathscr{X}$. Both spaces are copies of Euclidean n-space. To prescribe the target motion there is a Borel function

$$Y: T \times \mathscr{X} \to \mathscr{Y},$$

where T is an interval of real numbers containing 0 as its left-hand endpoint. The target motion is characterized by a stochastic parameter ξ which takes values in $\mathscr{X}$. If $\xi = x$, then $Y(t, x)$ gives the position of the target at time t.

The distribution of the parameter ξ is given by a probability density function f such that

$$\Pr\{\xi \in \mathscr{S}\} = \int_{\mathscr{S}} f(x)\,dx$$

for any Borel set $\mathscr{S} \subset \mathscr{X}$. Note that all integrations are taken with respect to Lebesgue measure on the indicated set.

Often we take $\mathscr{X} = \mathscr{Y}$ and consider ξ to give the position of the target at time 0. In this case the target motion is deterministic given knowledge of the target position at time 0, which has a probability distribution specified by f.

For $t \in T$, we shall use the notation Y_t for the transformation $Y(t, \cdot): \mathscr{X} \to \mathscr{Y}$. Let Y_t^i denote the ith component of Y_t and $\partial Y_t^i/\partial x_j$ denote the partial derivative of Y_t^i with respect to the jth component of x. For $t \in T$ we assume that Y_t has continuous first partial derivatives at all $x \in X$. Let $J(t, x)$ be the absolute value of the Jacobian of the transformation Y_t evaluated at x, i.e.,

$$J(t, x) = |\det(\partial Y_t^i(x)/\partial x_j)| \quad \text{for } x \in \mathscr{X} \text{ and } t \in T.$$

The transformations Y_t for $t \in T$ are assumed to have the following properties:

(a) Y_t is one-to-one for $t \in T$.

(b) $J(t, x)$ is assumed to be positive for all $x \in \mathscr{X}$ and $t \in T$.

In view of (a), we may define Y_t^{-1}, the inverse of Y_t, on range Y_t. The probability that the target is contained in a Borel set $\mathscr{S} \subset \mathscr{Y}$ at time t is given by

$$\int_{Y_t^{-1}(\mathscr{S})} f(x)\,dx.$$

3. Statement of problem. A *search plan* is a Borel function $\mu: T \times \mathscr{Y} \to [0, \infty)$ which may be thought of as specifying a density for search effort in both time and space. The integral $\int_0^t \mu(s, y)\,ds$ gives the search density applied at point y by time t and $\int_{\mathscr{Y}} \int_0^t \mu(s, y)\,ds\,dy$ gives the total search effort expended by time t.

Constraints are imposed on search effort by means of Borel functions $\varphi_1: T \times \mathscr{Y} \to [0, \infty]$, $\varphi_2: T \to [0, \infty]$ and $\varphi_3: T \to [0, \infty]$. We say that $\mu \in M(\varphi_1,$

$\varphi_2, \varphi_3)$ if and only if μ is a search plan and

(3.1a) $$\mu(t, y) \leqq \varphi_1(t, y) \quad \text{for } (t, y) \in T \times \mathscr{Y},$$

(3.1b) $$\int_{\mathscr{Y}} \mu(t, y)\, dy \leqq \varphi_2(t) \quad \text{for } t \in T,$$

(3.1c) $$\int_0^t \int_{\mathscr{Y}} \mu(t, y)\, dy\, dt \leqq \varphi_3(t) \quad \text{for } t \in T.$$

Condition (3.1a) limits the rate at which search density may be applied to a point. Intuitively, this bounds the rate at which search may be assigned to small areas. Condition (3.1b) constrains the rate at which search effort may be applied to the whole search space, and condition (3.1c) limits the total amount of search effort which may be applied by time t. If the constraint corresponding to φ_j is not imposed, we write $\varphi_j = \infty$. For convenience of notation, we let

$$M_{\varphi_1} = M(\varphi_1, \infty, \infty), \quad M_{\varphi_2} = M(\infty, \varphi_2, \infty), \quad M_{\varphi_3} = M(\infty, \infty, \varphi_3).$$

In order to evaluate the effectiveness of a search plan μ, we assume that there is a local effectiveness function $b:[0, \infty] \to [0, 1]$ such that for $x \in \mathscr{X}$ and $t \in T$,

$$b\left(\int_0^t \mu(s, Y_s(x))\, ds\right)$$

gives the probability of detecting the target by time t with plan μ given that $\xi = x$. In effect, the target is accumulating search density $\int_0^t \mu(s, Y_s(x))\, ds$ as it travels along its path during time $[0, t]$, and b gives the detection probability as a function of accumulated search density. For a search plan μ, we let

$$P_t[\mu] = \int_{\mathscr{X}} b\left(\int_0^t \mu(s, Y_s(x))\, ds\right) f(x)\, dx.$$

Then $P_t[\mu]$ gives the probability of detecting the target by time t using plan μ. Without extra mathematical difficulty (other than notational), one could assume that the function b depends on the parameter x. The results of this paper would still remain true upon making the obvious changes.

Suppose $\mu^* \in M(\varphi_1, \varphi_2, \varphi_3)$. If for some $t \in T$,

$$P_t[\mu^*] = \max\{P_t[\mu] : \mu \in M(\varphi_1, \varphi_2, \varphi_3)\},$$

then we say that μ^* is *t-optimal* within $M(\varphi_1, \varphi_2, \varphi_3)$. If μ^* is t-optimal for all $t \in T$, then we follow [1] in saying that μ^* is *uniformly optimal* within $M(\varphi_1, \varphi_2, \varphi_3)$.

If $T = [0, \infty)$, then for search plans μ we let

$$P_\infty[\mu] = \lim_{t \to \infty} P_t[\mu].$$

We say that μ^* is *optimal* within $M(\varphi_1, \varphi_2, \varphi_3)$ if

$$P_\infty[\mu^*] = \max\{P_\infty[\mu] : \mu \in M(\varphi_1, \varphi_2, \varphi_3)\}.$$

The problem of finding a t-optimal $\mu^* \in M(\varphi_1, \varphi_2, \varphi_3)$ is equivalent to the following problem.

Find a Borel function $\alpha: T \times \mathscr{X} \to [0, \infty)$ to maximize

$$\int_{\mathscr{X}} b\left(\int_0^t \alpha(s, x)\, ds\right) f(x)\, dx \tag{3.2}$$

subject to

$$\alpha(s, x) \leqq \varphi_1(s, Y_s(x)) \quad \text{for } s \in T \text{ and } x \in \mathscr{X}, \tag{3.3}$$

$$\int_{\mathscr{X}} \alpha(s, x) J(s, x)\, dx \leqq \varphi_2(s) \quad \text{for } s \in T, \tag{3.4}$$

$$\int_0^s \int_{\mathscr{X}} \alpha(u, x) J(u, x)\, dx\, du \leqq \varphi_3(s) \quad \text{for } s \in T. \tag{3.5}$$

To see the equivalence, we make the substitution

$$\begin{aligned} \mu(s, Y_s(x)) &= \alpha(s, x) \quad \text{for } s \in T, \quad x \in \mathscr{X}, \\ \mu(s, y) &= 0 \qquad\quad\;\; \text{for } y \notin \operatorname{rng} Y_s. \end{aligned} \tag{3.6}$$

Then (3.2) becomes $P_t[\mu]$. Using the above substitution and setting $y = Y_s(x)$, we see that (3.3) becomes (3.1a). Similarly (3.4) becomes (3.1b) by using a standard change of variable theorem (e.g., [2, p. 104]) as follows:

$$\int_{\mathscr{X}} \mu(s, Y_s(x)) \mathfrak{J}(s, x)\, dx = \int_{\mathscr{Y}} \mu(s, y)\, dy \leqq \varphi_2(s) \quad \text{for } s \in T,$$

and (3.5) is transformed to (3.1c). Thus, by use of (3.6) we see that a solution α to (3.2)–(3.5) yields a μ which is t-optimal within $M(\varphi_1, \varphi_2, \varphi_3)$, and similarly a plan μ which is t-optimal with $M(\varphi_1, \varphi_2, \varphi_3)$ yields a solution to (3.2)–(3.5).

The problem stated in (3.2)–(3.5) is difficult to solve in general because, for one thing, it is not separable. That is, in general one cannot express the objective functional (3.2) in the form

$$\int_T \int_{\mathscr{X}} \delta(t, x, \alpha(t, x))\, dx\, dt,$$

where $\delta: T \times \mathscr{X} \times [0, \infty) \to (-\infty, \infty)$, although the constraints (3.3)–(3.5) are separable. However, two important cases of the above problem which we can solve are presented in §§ 4 and 5. In both of these cases, the nature of the problem allows us to reduce it to a separable problem which we then solve by a Neyman–Pearson or Lagrange multiplier technique.

4. Optimal search plans in M_{φ_2} when target motion is factorable. We say that the target motion is *factorable* if in addition to the motion satisfying the assumptions of § 2, there exist Borel functions $j: \mathscr{X} \to (0, \infty)$ and $m: T \to (0, \infty)$ such that

$$J(t, x) = m(t) j(x) \quad \text{for } t \in T \text{ and } x \in \mathscr{X}.$$

We then say that the target motion is factorable with $J = mj$.

Theorem 4.1 shows that when target motion is factorable, the problem of finding the optimal search plan in M_{φ_2} for a target with conditionally deterministic motion may be reduced to solving a constrained separable optimization problem.

In the theorem we actually consider a somewhat larger class of plans than M_{φ_2}. Let $\tilde{M}_{\varphi_2}$ be the class of search plans μ such that

$$\int_{\mathscr{Y}} \mu(t, y)\, dy \leqq \varphi_2(t) \quad \text{for a.e. } t \in T,$$

where a.e. means almost every in Lebesgue measure. We note that any $\mu \in \tilde{M}_{\varphi_2}$ may be modified so that condition (3.1b) is satisfied for all $t \in T$ by letting $\mu(t, \cdot) = 0$ for those t for which (3.1b) is not satisfied. This modification makes μ a member of M_{φ_2} and does not change the value of $P_t[\mu]$ for any $t \in T$.

Let D be the class of Borel functions $d:\mathscr{X} \to [0, \infty)$. For $d \in D$, let

$$E(d) = \int_{\mathscr{X}} b(d(x)) f(x)\, dx, \qquad C(d) = \int_{\mathscr{X}} d(x) j(x)\, dx.$$

If $h:[0, \infty) \times \mathscr{X} \to (-\infty, \infty)$, we let $h'(\cdot, x)$ denote the derivative of $h(\cdot, x)$ for $x \in \mathscr{X}$.

THEOREM 4.1. *Let the target motion be factorable with $J = mj$. Suppose there exists a function $g:T \times \mathscr{X} \to [0, \infty)$ such that*

(i) *$g' \geqq 0$ and $g(t, x) = \int_0^t g'(s, x)\, ds$ for $t \in T, x \in \mathscr{X}$,*

(ii) *for all $t \in T$,*

$$C(g(t, \cdot)) = \Phi_2(t) \equiv \int_0^t \frac{\varphi_2(s)}{m(s)}\, ds,$$

$$E(g(t, \cdot)) = \max \{E(d): d \in D \text{ and } C(d) \leqq \Phi_2(t)\}. \tag{4.1}$$

Then for $t \in T$, μ^* defined by

$$\mu^*(t, y) = \begin{cases} g'(t, Y_t^{-1}(y)) & \text{for } y \in \text{range } Y_t, \\ 0 & \text{otherwise} \end{cases} \tag{4.2}$$

is uniformly optimal within $\tilde{M}_{\varphi_2}$. Moreover,

$$P_t[\mu^*] = \int_{\mathscr{X}} b(g(t, x)) f(x)\, dx \quad \text{for } t \in T. \tag{4.3}$$

Proof. We first note that (4.3) follows from $g'(t, x) = \mu^*(t, Y_t(x))$ for $t \in T$ and $x \in \mathscr{X}$. To see that $\mu^* \in \tilde{M}_{\varphi_2}$ we define

$$\varphi_2^*(t) = \int_{\mathscr{Y}} \mu^*(t, y)\, dy \quad \text{for } t \in T.$$

Using (4.2) and the change of variable $x = Y_t^{-1}(y)$, we have

$$\varphi_2^*(t) = \int_{\mathscr{Y}} g'(t, Y_t^{-1}(y))\, dy$$
$$= m(t) \int_{\mathscr{X}} g'(t, x) j(x)\, dx \quad \text{for } t \in T.$$

Thus

$$\int_0^t \frac{\varphi_2^*(s)}{m(s)}\,ds = \int_0^t \int_{\mathscr{X}} g'(s,x)j(x)\,dx\,ds$$

$$= \int_{\mathscr{X}} g(t,x)j(x)\,dx$$

$$= C(g(t,\cdot)) = \int_0^t \frac{\varphi_2(s)}{m(s)}\,ds \quad \text{for } t \in T.$$

It follows that $\varphi_2^*(t) = \varphi_2(t)$ for a.e. $t \in T$ and that $\mu^* \in \tilde{M}_{\varphi_2}$.

To see that μ^* is uniformly optimal, we suppose that μ^* fails to be t-optimal for a fixed $t \in T$. Hence, there is a $\mu \in \tilde{M}_{\varphi_2}$ such that

$$P_t[\mu] > P_t[\mu^*]. \tag{4.4}$$

For $s \in T$ and $x \in \mathscr{X}$, let

$$h(s,x) = \int_0^s \mu(u, Y_u(x))\,du$$

and recall that

$$g(s,x) = \int_0^s \mu^*(u, Y_u(x))\,du.$$

Then (4.4) becomes

$$E(h(t,\cdot)) > E(g(t,\cdot)). \tag{4.5}$$

However,

$$\begin{aligned} C(h(t,\cdot)) &= \int_{\mathscr{X}} \int_0^t \mu(s, Y_s(x))\,ds\, j(x)\,dx \\ &= \int_{\mathscr{X}} \int_0^t \frac{1}{m(s)} \mu(s, Y_s(x)) J(s,x)\,ds\,dx. \end{aligned} \tag{4.6}$$

Making the change of variable $y = Y_s(x)$, we have

$$\int_{\mathscr{X}} \mu(s, Y_s(x)) J(s,x)\,dx = \int_{\mathscr{Y}} \mu(s,y)dy \leqq \varphi_2(s) \quad \text{for a.e. } s \in T. \tag{4.7}$$

The last inequality follows from $\mu \in \tilde{M}_{\varphi_2}$. Interchanging the order of integration in (4.6) and using (4.7), we obtain

$$C(h(t,\cdot)) \leqq \int_0^t \frac{\varphi_2(s)}{m(s)}\,ds = \Phi_2(t). \tag{4.8}$$

By taking $d = h(t,\cdot)$, we see that (4.5) and (4.8) together contradict (4.1), which is true by assumption. This shows that μ^* is uniformly optimal and finishes the proof.

We now discuss methods which may be used to find the function g in Theorem 1.

Let b' denote the derivative of the restriction of b to $[0, \infty)$. Suppose that b satisfies the following conditions:

(4.9a) b' is positive (possibly $+\infty$), continuous, and strictly decreasing;

(4.9b) $b(0) = 0$ and $0 \leqq b(z) \leqq 1$ for $z \geqq 0$.

Define, for $x \in \mathscr{X}$ and $z \geqq 0$,

$$\rho(z, x) = f(x)b'(z)/j(x),$$

and let $\rho^{-1}(\cdot, x)$ be the inverse function for $\rho(\cdot, x)$. For convenience, we extend the domain of $\rho^{-1}(\cdot, x)$ by letting $\rho^{-1}(\gamma, x) = 0$ for $\gamma > \rho(0, x)$. In case $f(x) = 0$, we take $\rho^{-1}(\gamma, x) = 0$ for $\gamma > 0$. For $\gamma > 0$, let

$$A(\gamma) = \int_{\mathscr{X}} \rho^{-1}(\gamma, x) j(x)\, dx.$$

THEOREM 4.2. *Let the target motion be factorable with $J = mj$ and let b satisfy conditions* (4.9a, b). *Then A has an inverse $A^{-1}: (0, \infty) \to [0, \infty)$ such that*

$$A(A^{-1}(t)) = t \quad \textit{for } 0 \leqq t < \infty. \tag{4.10}$$

Moreover, a function g satisfying assumption (ii) *of Theorem* 4.1 *is given by*

$$g(t, x) = \rho^{-1}(A^{-1}(\Phi_2(t)), x) \quad \textit{for } t \in T, \quad x \in \mathscr{X}. \tag{4.11}$$

Proof. The proof of this theorem is based on a modification of the proof of Theorem 2 of [9].

We first observe that by virtue of conditions (4.9),

$$1 \geqq b(z) = \int_0^z b'(r)\, dr \geqq z b'(z) \quad \text{for } 0 \leqq z < \infty.$$

Hence $b'(z) \leqq 1/z$ which implies that $b'^{-1}(\gamma) \leqq 1/\gamma$ for $\gamma > 0$, where b'^{-1} is the inverse of b'. Thus

$$\rho^{-1}(\gamma, x) \leqq \frac{f(x)}{j(x)\gamma} \quad \text{for } \gamma > 0, \quad x \in \mathscr{X},$$

and

$$A(\gamma) = \int_{\mathscr{X}} \rho^{-1}(\gamma, x) j(x)\, dx \leqq \frac{1}{\gamma} \int_{\mathscr{X}} f(x)\, dx = \frac{1}{\gamma}.$$

Since $\rho^{-1}(\cdot, x)$ is continuous and strictly decreasing for $\gamma \leqq f(x)b'(0)/j(x)$, one may use the monotone convergence theorem to show that A is continuous and strictly decreasing until it takes the value zero. One can also show that $A(\gamma) \to \infty$ as $\gamma \to 0$. Thus A^{-1} satisfying (4.10) exists.

Let g be defined by equation (4.11). It follows from (4.10) and the definition of A that

$$\int_{\mathscr{X}} g(t, x) j(x)\, dx = \int_{\mathscr{X}} \rho^{-1}(A^{-1}(\Phi_2(t)), x) j(x)\, dx = \Phi_2(t) \quad \text{for } t \geqq 0.$$

Let

$$e(z,x) = b(z)f(x),$$
$$c(z,x) = zj(x) \quad \text{for } z \geqq 0, \quad x \in \mathscr{X}.$$

Note that $e'(z,x) = b'(z)f(x)$ and $c'(z,x) = j(x)$, for $z \geqq 0$ and $x \in \mathscr{X}$. Letting $\lambda = A^{-1}(\Phi_2(t))$, we see by (4.11) and the decreasing nature of b' that

$$\begin{aligned} e'(z,x) &\geqq \lambda c'(z,x) \quad \text{for } 0 \leqq z < g(t,x), \\ e'(z,x) &\leqq \lambda c'(z,x) \quad \text{for } g(t,x) < z \quad \text{for } t \in T, \quad x \in \mathscr{X}. \end{aligned} \tag{4.12}$$

The inequalities in (4.12) are a derivative form of the Neyman–Pearson sufficient conditions for constrained maximization of a nonlinear separable functional. Since $\lambda \geqq 0$, we have by [11, Thm. 1] (also quoted as [9, Thm. 1]) that g defined by (4.11) satisfies assumption (ii) of Theorem 4.1 for each $t \in T$. This proves the theorem.

In practice, one may have to calculate A numerically and find its inverse by numerical or graphical techniques. Observe that $\rho^{-1}(\cdot, x)$ is continuous and decreasing and that A^{-1} is also continuous and decreasing. Since Φ_2 is continuous and increasing, it follows that $g(\cdot, x)$ defined by (4.11) is continuous and increasing. Moreover, one may check that $g(0,x) = 0$, since $\Phi(0) = 0$. Thus in most cases covered by Theorem 4.2, we will have that $g'(\cdot, x)$ exists, is nonnegative, and

$$\int_0^t g'(s,x)\,ds = g(t,x) \quad \text{for } t \in T. \tag{4.13}$$

That is, both assumptions (i) and (ii) of Theorem 4.1 will be satisfied, and we may use Theorem 4.1 to find μ^*.

Let b'' denote the derivative of b'. If in addition to conditions (4.9a, b), we require that

$$b'(z) < \infty \quad \text{for } z \in [0, \infty),$$
$$b'' \text{ is continuous on } [0,\infty) \quad \text{and} \quad b''(0) < 0,$$

and if

$$\Gamma \equiv \operatorname{ess\,sup} f < \infty,$$

then it is shown in [8] that g defined by (4.11) satisfies (4.13). We summarize this in the following theorem.

THEOREM 4.3. *Let the target motion be factorable with $J = mj$ and $\Gamma < \infty$. Assume that*

(a) $b(0) = 0$ *and* $0 \leqq b(z) \leqq 1$ *for* $z \geqq 0$,
(b) $b'(z)$ *is finite, positive, continuous and strictly decreasing for* $z \in [0, \infty)$,
(c) b'' *is continuous on* $[0, \infty)$ *and* $b''(0) < 0$.

Then μ^* defined by, for $t \in T$,

$$\mu^*(t,y) = \begin{cases} g'(A^{-1}(\Phi_2(t)), Y_t^{-1}(y)) & \textit{for } y \in \text{range } Y_t, \\ 0 & \textit{otherwise}, \end{cases}$$

is uniformly optimal within $\tilde{M}_{\varphi_2}$, where g is given by (4.11).

Example 4.4. As an example of the use of Theorems 4.1 and 4.2, we find an optimal search plan for a specific moving target problem.

Let $\mathscr{X} = \mathscr{Y}$ be Euclidean 2-space and the parameter ξ be the target's position at time 0. We suppose that ξ has a circular normal probability distribution with parameter $\sigma > 0$, i.e.,

$$f(x) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{x_1^2 + x_2^2}{2\sigma^2}\right) \quad \text{for } x \in \mathscr{X}. \tag{4.14}$$

Let $T = [0, \infty)$. The target's velocity is assumed to be constant, with its speed proportional to its distance from the origin at time 0 and its heading given by the vector from the origin to its position at time 0. Let v be the proportionality constant for the target speed. Then

$$Y(t, x) = (1 + tv)(x_1, x_2) \quad \text{for } t \in T, \quad x \in \mathscr{X},$$

where scalar multiplication of a vector is understood in the usual manner. It is easy to calculate that

$$J(t, x) = (1 + tv)^2 \quad \text{for } t \in T \quad \text{and} \quad x \in \mathscr{X},$$

so that the target motion is factorable with $j(x) = 1$ for $x \in \mathscr{X}$ and $m(t) = (1 + tv)^2$ for $t \in T$. Recall that φ_2 specifies the rate at which search effort may be exerted, and

$$\Phi_2(t) = \int_0^t \frac{\varphi_2(s)}{m(s)}\, ds = \int_0^t \frac{\varphi_2(s)}{(1 + sv)^2}\, ds.$$

We assume φ_2 is continuous so that Φ_2', the derivative of Φ_2, is given by

$$\Phi_2'(t) = \varphi_2(t)/(1 + tv)^2 \quad \text{for } t \geqq 0.$$

The local effectiveness function b is assumed to be given by

$$b(z) = 1 - e^{-z} \quad \text{for } z \geqq 0.$$

We shall find the optimal search plan μ^* by using Theorem 4.2. To do this, we calculate the function A of that theorem. Observe that

$$\rho(z, x) = \frac{e^{-z}}{2\pi\sigma^2} \exp\left(-\frac{x_1^2 + x_2^2}{2\sigma^2}\right) \quad \text{for } z \geqq 0, \quad x \in \mathscr{X},$$

and

$$\rho^{-1}(\gamma, x) = \left[-\ln(2\pi\sigma^2\gamma) - \frac{x_1^2 + x_2^2}{2\sigma^2}\right]^+, \tag{4.15}$$

where $s^+ = \max\{0, s\}$ for all real numbers s.

Thus

$$\begin{aligned} A(\gamma) &= \int_{\mathscr{X}} \rho^{-1}(\gamma, x)\, dx \\ &= 2\pi \int_0^\infty [-\ln(2\pi\sigma^2\gamma) - r^2/2\sigma^2]^+ r\, dr, \end{aligned}$$

where we have introduced polar coordinates in the second integral. It is easy to calculate that

$$A(\gamma) = \pi\sigma^2(\ln(2\pi\sigma^2\gamma))^2 \quad \text{for } \gamma \leqq (2\pi\sigma^2)^{-1}, \tag{4.16}$$

and

$$A^{-1}(t) = \frac{1}{2\pi\sigma^2}\exp\left[-\left(\frac{t}{\pi\sigma^2}\right)^{1/2}\right] \quad \text{for } t \geqq 0. \tag{4.17}$$

Substituting (4.15) and (4.17) into (4.11), we find that for $t \in T$ and $x \in \mathscr{X}$,

$$g(t,x) = \left[\left(\frac{\Phi_2(t)}{\pi\sigma^2}\right)^{1/2} - \frac{x_1^2 + x_2^2}{2\sigma^2}\right]^+,$$

$$g'(t,x) = \begin{cases} \dfrac{\varphi_2(t)}{2\sigma(\pi\Phi_2(t))^{1/2}(1+tv)^2} & \text{for } x_1^2 + x_2^2 \leqq 2\sigma^2\left(\dfrac{\Phi_2(t)}{\pi\sigma^2}\right)^{1/2}, \\ 0 & \text{otherwise.} \end{cases} \tag{4.18}$$

Clearly $g(\cdot, x)$ satisfies (4.13), so that by Theorems 4.1 and 4.2, we have for $t \geqq 0$, $x \in \mathscr{X}$,

$$\mu^*(t,x) = \begin{cases} \dfrac{\varphi_2(t)}{\pi R^2(t)} & \text{for } x_1^2 + x_2^2 \leqq R^2(t), \\ 0 & \text{otherwise,} \end{cases} \tag{4.19}$$

where

$$R^2(t) = 2\sigma^2(1+tv)^2\left(\frac{\Phi_2(t)}{\pi\sigma^2}\right)^{1/2} \quad \text{for } t \geqq 0.$$

Thus $R(t)$ gives the search radius by time t, and the search plan μ^* calls for having $\mu^*(t,\cdot)$ constant over the disc of radius $R(t)$ centered at the origin. Because this disc expands in time, cumulative effort tends to be heavier near the origin.

We now calculate $P_t[\mu^*]$ for $t \geqq 0$ by use of (4.3) and (4.18) as follows:

$$\begin{aligned} P_t[\mu^*] &= \int_{\mathscr{X}} [1 - e^{-g(t,x)}]\frac{1}{2\pi\sigma^2}\exp\left(-\frac{x_1^2+x_2^2}{2\sigma^2}\right)dx \\ &= 1 - \frac{1}{\sigma^2}e^{-Q^2(t)}\int_0^{\sqrt{2}\sigma Q(t)} r\,dr - \frac{1}{\sigma^2}\int_{\sqrt{2}\sigma Q(t)}^{\infty} r\exp\left(\frac{-r^2}{2\sigma^2}\right)dr, \end{aligned}$$

where $Q^2(t) = (\Phi_2(t)/\pi\sigma^2)^{1/2}$ and we have transformed the integral to polar coordinates. Thus

$$P_t[\mu^*] = 1 - \exp\left[-\left(\frac{\Phi_2(t)}{\pi\sigma^2}\right)^{1/2}\right]\left[1 + \left(\frac{\Phi_2(t)}{\pi\sigma^2}\right)^{1/2}\right] \quad \text{for } t \geqq 0. \tag{4.20}$$

Consider the special case of the above search where $v = 0$. Then the target is stationary and

$$g(t,x) = \int_0^t \mu^*(s,x)\,ds$$

is the search density placed at x by time t. In this case, $g(t, \cdot)$ becomes the optimal distribution of $\Phi_2(t)$ amount of search effort, as given by Koopman in [6].

It is obvious from (4.20) that the probability of detecting this moving target will go to unity if and only if $\Phi_2(t) \to \infty$ as $t \to \infty$. For example, if $\varphi_2(t) = U$ for $t \geqq 0$, we have

$$\Phi_2(t) = \int_0^t \frac{U}{(1 + sv)^2} ds = Ut(1 + tv)^{-1}$$

and

$$\lim_{t \to \infty} P_t[\mu^*] = 1 - \exp\left[- \left(\frac{U}{v\pi\sigma^2}\right)^{1/2}\right]\left[1 + \left(\frac{U}{v\pi\sigma^2}\right)^{1/2}\right] < 1.$$

We note that the analysis leading to (4.19) and (4.20) does not depend on the fact that $m(t) = (1 + tv)^2$ for $t > 0$. In fact, these equations remain true for arbitrary continuous $m > 0$ provided we take

$$R^2(t) = 2\sigma^2 m(t)\left(\frac{\Phi_2(t)}{\pi\sigma^2}\right)^{1/2} \quad \text{for } t > 0,$$

and remember that

$$\Phi_2(t) = \int_0^t \frac{\varphi_2(s)}{m(s)} ds \quad \text{for } t \geqq 0.$$

Example 4.5. As a second example, we shall use Theorems 4.1 and 4.2 to find the optimal allocation of distributed sensor fields.

Consider a situation in which the target has previously been detected and then contact is lost. A search for the target commences after a time delay t_l from the loss of contact. The target is assumed to move radially from the last point of contact at a constant speed which is obtained from a uniform distribution on $[v_1, v_2]$. Similarly the target's course is assumed to be constant and to be obtained from a uniform distribution on $[\theta_1, \theta_2]$.

Let us take $t = 0$ to be the time at which search commences and choose the origin of the coordinate system to be the target's position at the point of last contact. At $t = 0$, the target is located in the region X_0 shown in Fig. 1. We take $r_1 = v_1 t_l$ and $r_2 = v_2 t_l$.

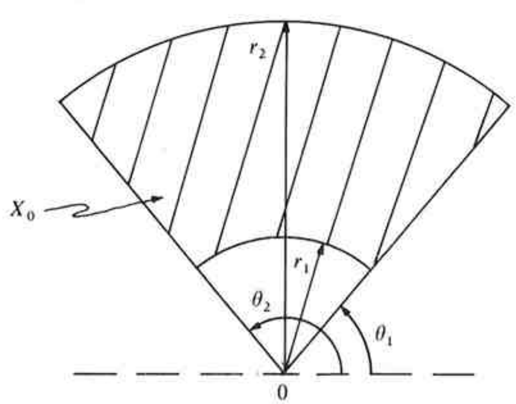

FIG 1. *Region in which target is located at* $t = 0$

The density f for the target location distribution at $t = 0$, is

$$f(x) = \begin{cases} [(r_2 - r_1)(\theta_2 - \theta_1)(x_1^2 + x_2^2)^{1/2}]^{-1} & \text{for } x \in X_0, \\ 0 & \text{otherwise}. \end{cases}$$

Here angles are measured in radians. The target motion is given by

$$Y_t(x) = \left(\frac{t + t_l}{t_l}\right)(x_1, x_2) \quad \text{for } x \in X, \quad t \geqq 0.$$

Clearly,

$$J(x, t) = \left(\frac{t + t_l}{t_l}\right)^2 \quad \text{for } x \in X, \quad t \geqq 0,$$

and the target motion is factorable with $j(x) = 1$ for $x \in X$, $m(t) = [(t + t_l)/t_l]^2$ for $t \geqq 0$.

We take $b(z) = 1 - e^{-z}$ for $z > 0$. This form for b arises from assuming that if one has N sensors uniformly distributed in a region of area A' and operating for a time period S, then the probability of detecting the target, given it is in the region during S, is $1 - \exp(-\beta NS/A')$ for some constant $\beta > 0$. Thus the effort density which accumulates at points in this region during S is given by $z = \beta NS/A'$. If M is the total number of sensors which can be usefully employed at one time, then search effort is applied at the constant rate $\varphi_2(t) = U = \beta M$. Thus

$$\Phi_2(t) = \int_0^t \frac{\varphi_2(s)}{m(s)}\, ds = Ut_l\left(\frac{t}{t + t_l}\right) \quad \text{for } t \geqq 0.$$

For this example, the detection function is independent of the speed of the target. However, speed could be accounted for by allowing b to depend on x, e.g., $b(x, z) = 1 - \exp(-k(x)z)$.

In order to use Theorem 4.2, we calculate

$$\rho(z, x) = f(x)b(z) \quad \text{for } x \in X, \quad z \geqq 0,$$

$$\begin{aligned} \rho^{-1}(\gamma, x) &= [\ln(f(x)/\gamma)]^+ \\ &= \{-\ln[(r_2 - r_1)(\theta_2 - \theta_1)(x_1^2 + x_2^2)^{1/2}\gamma]\}^+, \quad x \in X, \quad \gamma > 0. \end{aligned}$$

Let $K(\gamma) = [r_2(r_2 - r_1)(\theta_2 - \theta_1)\gamma]^{-1}$ for $\gamma > 0$. Then

$$\begin{aligned} A(\gamma) &= \int_{X_0} \rho^{-1}(\gamma, x)\, dx = \int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2} [\ln(r_2K(\gamma)/r)]^+ r\, dr\, d\theta \\ &= \frac{r_1^2(\theta_2 - \theta_1)}{2}\left\{\frac{1}{2}[(r_2K(\gamma)/r_1)^2 - 1] - \ln(r_2K(\gamma)/r_1)\right\} \quad \text{for } \frac{r_1}{r_2} \leqq K(\gamma) \leqq 1 \end{aligned}$$

and

$$A(\gamma) = \frac{(\theta_2 - \theta_1)}{2}\left\{\frac{1}{2}(r_2^2 - r_1^2) + r_2^2\ln(K(\gamma)) - r_1^2\ln(r_2K(\gamma)/r_1)\right\} \quad \text{for } K(\gamma) \geqq 1.$$

In order to find μ^*, it remains to calculate A^{-1} and to make use of (4.2) and (4.11). Observe that (4.11) yields

$$g(t, x) = \{\ln(f(x)) - \ln[A^{-1}(\Phi_2(t))]\}^+.$$

From this it follows that $g'(t, \cdot)$ is equal to a constant inside the region $R_t = \{x : f(x) > A^{-1}(\Phi_2(t))\}$ and 0 outside of R_t. Now from (4.2) we see that $\mu^*(t, \cdot)$ is constant over the region $Z(t) = Y_t(R_t)$ and 0 outside $Z(t)$. Thus, in order to find the optimal plan μ^*, we need only determine $Z(t)$ and apply effort density at a uniform rate over $Z(t)$. This behavior is a property of the exponential detection function. That is, $\mu^*(t, \cdot)$ is equal to a constant over the region in which search is applied at time t. If one can calculate the area $a(t)$ of $Z(t)$, it follows that

$$\mu^*(t, x) = \begin{cases} \varphi_2(t)/a(t) & \text{for } x \in Z(t), \\ 0 & \text{otherwise.} \end{cases}$$

In view of the above discussion, we calculate μ^* as follows. Let K be the solution of

$$\frac{2\Phi_2(t)}{r_1^2(\theta_2 - \theta_1)} = \frac{1}{2}[(r_2K/r_1)^2 - 1] - \ln(r_2K/r_1).$$

If $K \leqq 1$, then

$$\mu^*(t, (r, \theta)) = \begin{cases} \dfrac{U}{a(t)} & \text{for } \theta_1 \leqq \theta \leqq \theta_2, \quad (t + t_l)v_1 \leqq r \leqq (t + t_l)Kv_2, \\ 0 & \text{otherwise,} \end{cases}$$

where $a(t) = \frac{1}{2}(\theta_2 - \theta_1)(t + t_l)^2[(Kv_2)^2 - v_1^2]$.

If $K \geqq 1$, then

$$\mu^*(t, (r, \theta)) = \begin{cases} \dfrac{U}{a(t)} & \text{for } \theta_1 \leqq \theta \leqq \theta_2, \quad (t + t_l)v_1 \leqq r \leqq (t + t_l)v_2, \\ 0 & \text{otherwise.} \end{cases}$$

where $a(t) = \frac{1}{2}(\theta_2 - \theta_1)(t + t_l)^2(v_2^2 - v_1^2)$.

The region in which the target may be located expands as time increases, and from the above we see that the initial phase of the search covers only part of this region. In fact, if

$$\frac{2U}{t_l} \leqq (\theta_2 - \theta_1)[\tfrac{1}{2}(v_2^2 - v_1^2) - v_1^2 \ln(v_2/v_1)],$$

then the search never expands to cover the entire region in which the target may be located. One may also show that if search effort is applied at a constant rate U, then the probability of detection will always remain less than 1.

The optimal search plan calls for spreading effort or sensors uniformly over a region which is continuously increasing. In practice, one would have to approximate this by choosing a time increment, possibly the lifetime of a sensor, and spreading sensors uniformly over a given region for that increment of time. The given region would, of course, be chosen to approximate the region searched by the optimal plan in that increment.

5. Plans which are optimal in $M(\varphi_1, \infty, l)$. For the special case in which $T = [0, \infty)$, $\varphi_3 = l$, a constant, and only constraints (3.1a) and (3.1c) are imposed, we present a method of finding the optimal plan within $M(\varphi_1, \infty, l)$ by making use of an obvious generalization of Theorem 3.2 of [10].

Plans in $M(\varphi_1, \infty, l)$ satisfy a constraint l on the total effort used and a constraint φ_1 on the search density μ, but no additional constraint is imposed on the rate at which search effort is applied to the total search area.

By our discussion in § 3, we seek a Borel function $\alpha:[0, \infty) \times \mathscr{X} \to [0, \infty)$ which maximizes

$$\int_{\mathscr{X}} b\left(\int_0^\infty \alpha(s, x)\, ds\right) f(x)\, dx \tag{5.1}$$

subject to

$$\alpha(s, x) \leqq \varphi_1(s, Y_s(x)) \quad \text{for } s \in T, \quad x \in \mathscr{X}, \tag{5.2a}$$

$$\int_{\mathscr{X}} \int_0^\infty \alpha(s, x) J(s, x)\, ds\, dx \leqq l. \tag{5.2b}$$

Let Ψ be the class of Borel functions $\psi:[0, \infty) \times \mathscr{X} \to [0, \infty)$ such that $\psi(s, x) \leqq \varphi_1(s, Y_s(x))$ for $s \geqq 0$ and $x \in \mathscr{X}$. Define, for $\psi \in \Psi$,

$$\tilde{E}(\psi) = \int_{\mathscr{X}} b\left(\int_0^\infty \psi(s, x)\, ds\right) f(x)\, dx,$$

$$\tilde{C}(\psi) = \int_{\mathscr{X}} \int_0^\infty \psi(s, x) J(s, x)\, ds\, dx.$$

Thus our problem is to find $\psi \in \Psi$ such that $\tilde{C}(\psi^*) = l$ and

$$\tilde{E}(\psi^*) = \max\{\tilde{E}(\psi): \psi \in \Psi \text{ and } \tilde{C}(\psi) \leqq l\}. \tag{5.3}$$

The solution to (5.3) may be approached by defining an auxiliary function $\tilde{e}$ as follows: for $x \in \mathscr{X}$ and $q \geqq 0$, let

$$\tilde{e}(q, x) = \sup\left\{f(x) b\left(\int_0^\infty \psi(s, x)\, ds\right) : \int_0^\infty \psi(s, x) J(s, x)\, ds = q \text{ and } \psi(s, x) \leqq \varphi_1(s, Y_s(x)) \text{ for } s \geqq 0\right\}.$$

One may think of $\tilde{e}(x, q)$ as giving the maximum effectiveness which can be obtained at x for the point cost q. Suppose that for $x \in \mathscr{X}$ and cost q, we can find a function $\psi_q(\cdot, x):[0, \infty) \to [0, \infty)$ such that

$$\tilde{e}(q, x) = f(x) b\left(\int_0^\infty \psi_q(s, x)\, ds\right),$$

$$q = \int_0^\infty \psi_q(s, x) J(s, x)\, ds.$$

Then we have reduced our problem to the following separable problem: find $h:\mathscr{X} \to [0, \infty)$ such that

$$\int_{\mathscr{X}} \tilde{e}(h(x), x)\, dx = \sup\left\{\int_{\mathscr{X}} \tilde{e}(p(x), x)\, dx : p \geqq 0 \text{ and } \int_{\mathscr{X}} p(x)\, dx \leqq l\right\}$$

and

$$\int_{\mathscr{X}} h(x)\,dx = l.$$

For then $\alpha(t, x) = \psi_{h(x)}(t, x)$ for $t \geqq 0$, and $x \in \mathscr{X}$ solves (5.1) and (5.2). This is the idea behind the following theorem.

THEOREM 5.1. *Let $\psi^* \in \Psi$ be such that $\tilde{C}(\psi^*) = l$ and for $x \in \mathscr{X}$,*

$$\tilde{e}(h(x), x) = f(x)b\left(\int_0^\infty \psi^*(s, x)\,ds\right), \tag{5.4}$$

where

$$h(x) = \int_0^\infty \psi^*(s, x)J(s, x)\,ds. \tag{5.5}$$

If there exists a $\lambda \geqq 0$ such that

$$\tilde{e}(h(x), x) - \lambda h(x) = \max\{\tilde{e}(z, x) - \lambda z : z \geqq 0\} \quad \textit{for } x \in \mathscr{X}, \tag{5.6}$$

then μ^* defined by

$$\mu^*(s, y) = \begin{cases} \psi^*(s, Y_s^{-1}(y)) & \textit{for } s \geqq 0 \textit{ and } y \in \text{range } Y_s, \\ 0 & \textit{otherwise}, \end{cases}$$

is optimal in $M(\varphi_1, \infty, l)$.

Proof. Let $\mu \in M(\varphi_1, \infty, l)$ and define

$$\begin{aligned} \psi(s, x) &= \mu(s, Y_s(x)) \quad \text{for } s \geqq 0, \quad x \in \mathscr{X}, \\ q(x) &= \int_0^\infty \psi(s, x)J(s, x)\,ds \quad \text{for } x \in \mathscr{X}. \end{aligned} \tag{5.7}$$

Then by (5.6) and the definition of $\tilde{e}$,

$$\begin{aligned} &\tilde{e}(h(x), x) - f(x)b\left(\int_0^\infty \mu(s, Y_s(x))\,ds\right) \\ &\qquad \geqq \tilde{e}(h(x), x) - \tilde{e}(q(x), x) \geqq \lambda(h(x) - q(x)) \quad \text{for } x \in \mathscr{X}. \end{aligned}$$

Thus

$$\int_{\mathscr{X}} \tilde{e}(h(x), x)\,dx - \tilde{E}(\psi) \geqq \lambda\left(\int_{\mathscr{X}} h(x)\,dx - \int_{\mathscr{X}} q(x)\,dx\right). \tag{5.8}$$

By (5.4) and (5.5), we have $\int_{\mathscr{X}} \tilde{e}(h(x), x)\,dx = E(\psi^*)$ and $\int_{\mathscr{X}} h(x)\,dx = \tilde{C}(\psi^*) = l$. By (5.7) and the fact that $\mu \in M(\varphi_1, \infty, l)$, we have

$$\begin{aligned} \int_{\mathscr{X}} q(x)\,dx = \int_{\mathscr{X}} \int_0^\infty \psi(s, x)J(s, x)\,ds\,dx &= \int_{\mathscr{X}} \int_0^\infty \mu(s, Y_s(x))J(s, x)\,ds\,dx \\ &= \int_{\mathscr{Y}} \int_0^\infty \mu(s, y)\,ds\,dy \leqq l. \end{aligned}$$

Thus the inequality in (5.8) becomes $\tilde{E}(\psi^*) - \tilde{E}(\psi) \geqq 0$, i.e., $P_\infty[\mu^*] \geqq P_\infty[\mu]$, and the theorem is proved.

Example 5.2. We now find an optimal search plan by use of Theorem 5.1. Let $\mathscr{X}$ and $\mathscr{Y}$ be copies of Euclidean 2-space. In this case, the parameter space $\mathscr{X}$ gives the target's velocity. The target starts at the origin at time 0 and its velocity vector is chosen from a circular normal distribution with density given by (4.14). Thus

$$Y_s(x) = s(x_1, x_2)$$

and

$$J(s, x) = s^2 \quad \text{for } x \in \mathscr{X} \text{ and } s \geqq 0.$$

Let $\varphi_1 = k$, where k is a positive number, and define

$$b(z) = 1 - e^{-z^3} \quad \text{for } z \geqq 0. \tag{5.9}$$

If for $x \in \mathscr{X}$, we let

$$\psi_t(s, x) = \begin{cases} k & \text{for } 0 \leqq s \leqq t, \\ 0 & \text{otherwise,} \end{cases}$$

then the resulting cost at $x \in \mathscr{X}$ is $\int_0^\infty \psi_t(s, x) J(s, x)\, ds = kt^3/3$. Because of the strictly increasing nature of $J(\cdot, x)$, it is clear that $\psi_t(\cdot, x)$ gives the least cost method of applying kt "effort" at $x \in \mathscr{X}$. Thus

$$\tilde{e}(q, x) = f(x) b\left(\int_0^{(3q/k)^{1/3}} k\, ds\right) = f(x)(1 - e^{-3k^2 q}).$$

Let

$$h_\lambda(x) = \frac{1}{3k^2}\left[-\ln\left(\frac{2\pi\sigma^2\lambda}{3k^2}\right) - \frac{(x_1^2 + x_2^2)}{2\sigma^2}\right]^+ \quad \text{for } x \in \mathscr{X} \text{ and } \lambda > 0.$$

One may check that by taking $h = h_\lambda$, (5.6) is satisfied for any $\lambda > 0$. By a straightforward calculation similar to that which led to (4.16), we find that

$$\int_{\mathscr{X}} h_\lambda(x)\, dx = \frac{\pi\sigma^2}{3k^2} \ln^2\left(\frac{2\pi\sigma^2\lambda}{3k^2}\right).$$

Let $Q = \sigma^{-1} k (3l/\pi)^{1/2}$; then taking

$$\lambda = \frac{3k^2 \exp[-Q]}{2\pi\sigma^2},$$

we see that $C(h_\lambda) = l$ and

$$h_\lambda(x) = \frac{1}{3k^2}\left[Q - \frac{(x_1^2 + x_2^2)}{2\sigma^2}\right]^+ \quad \text{for } x \in \mathscr{X}.$$

For $x \in \mathscr{X}$, let

$$\psi^*(s, x) = \begin{cases} k & \text{for } 0 \leqq s \leqq k^{-1}\left\{\left[Q - \dfrac{(x_1^2 + x_2^2)}{2\sigma^2}\right]^+\right\}^{1/3}, \\ 0 & \text{otherwise.} \end{cases}$$

Then we see that ψ^* satisfies the conditions of Theorem 5.1, and thus μ^* defined by

$$(5.10)\quad \mu^*(s, y) = \begin{cases} k & \text{for } y_1^2 + y_2^2 \leqq 2\sigma^2 s^2(Q - k^3 s^3), \\ 0 & \text{otherwise,} \end{cases} \qquad \text{for } y \in \mathscr{Y}, \quad s \geqq 0,$$

is optimal within $M(k, \infty, l)$.

It is interesting to note that μ^* has the property that search begins at the origin expanding in a circle until a time at which it reaches a maximum distance from the origin. At this time, the search circle begins to shrink back to the origin! An examination of (5.14) reveals the reason for this behavior. At time s, the optimal plan calls for searching the region of possible target positions which would result from restricting the target's speed to lie between 0 and $V(s)$, where $V(s) = [\sigma(2Q - 2k^3s^3)^{1/2}]^+$ for $s \geqq 0$. Note that V is a monotonic decreasing function.

The choice of b in (5.9) is rather unnatural, but it allows us to calculate μ^* analytically. If the usual negative exponential local effectiveness function is used, one can still find μ^*, but numerical methods are required. However, without actually calculating the optimal allocation, one can show that it calls for placing some search effort at each point in the plane, the points farthest from the origin receiving search effort for the shortest time. In this case $\mu^*(0, y) = k$ for $y \in \mathscr{Y}$, while for $t > 0$, $\mu^*(t, y)$ equals k inside a circle and zero outside. This circle shrinks down to the origin by a finite time.

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