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SEARCH FOR TARGETS WITH GENERALIZED CONDITIONALLY DETERMINISTIC MOTION*

LAWRENCE D. STONE†

Abstract. A generalized version of the conditionally deterministic motion defined by Stone and Richardson in [8 (this Journal, 1974)] is considered in which the target motion takes place in Y, a subset of Euclidean n-space, and depends on a parameter ξ which takes values in $X \times \Omega$, where X is a subset of Euclidean n-space and Ω is a parameter set. (In [8], the parameter ξ is required to take values in X). This generalization is very similar to that considered by Pursiheimo in [5 (this Journal, 1977)]. There is a function $\eta: X \times \Omega \times [0, T] \to Y$ that prescribes the target motion in the sense that if $\xi = (x, \omega)$, then $\eta(x, \omega, t)$ gives the target's position for $0 \leqq t \leqq T$. The parameter ξ is assumed to be a random variable whose distribution is known to the searcher.

A search plan density is specified by a function $\psi: Y \times [0, T] \to [0, \infty)$ which is a density in time and space for search effort. There is a function m which constrains the rate at which search effort may be applied to the search space. That is, search plan densities are required to satisfy $\int_Y \psi(y, t)\, dy \leqq m(t)$ for $0 \leqq t \leqq T$. In addition, the densities are required to be bounded by a constant B. In Theorem 4.1, sufficient conditions are found for a search plan density to maximize the probability of detecting the target by time T. Theorem 5.1 finds necessary conditions for a plan to maximize probability of detection by time T. The case of unbounded densities is covered as the special case where $B = \infty$.

1. Introduction. In this paper we consider search for a target whose motion is a generalized version of conditionally deterministic motion as considered in [8], [7] and [5]. The target is assumed to be moving in a subset Y of Euclidean n-space, $\mathscr{E}_n$. Its motion is determined by a target motion function $\eta: X \times \Omega \times [0, T] \to Y$ and a stochastic parameter ξ. The parameter ξ takes values in $X \times \Omega$ where X is also a subset of $\mathscr{E}_n$ and Ω is a parameter set. Specifically, if $\xi = (x, \omega)$, then $\eta(x, \omega, t)$ gives the target's position at time t for t in the interval $[0, T]$. However, the value of ξ is not known to the search planner; only the distribution of ξ is known. In [8], the stochastic parameter ξ was assumed to take values in X, and ξ was required to have the same dimension as Y. In contrast, the generalized motion assumed here allows one to consider problems where the dimension of ξ is larger than that of Y. This makes it possible to consider problems where both the target's initial position and velocity are random.

As an example of a problem involving this generalized conditionally deterministic motion, consider the search for a fleeing target discussed in [4, p. 16]. In this problem the target's distribution at time 0 is circular normal and the target is assumed to be moving with a constant speed and direction. The speed is assumed to be known to the searcher, but the direction is assumed to be unknown and to have a uniform distribution over $[0, 2\pi]$. Since the target's speed is assumed known, specification of an initial position x and a direction θ completely determines the target's path. Thus, the target's motion is a generalized conditionally deterministic one with $Y = X = \mathscr{E}_2$, the plane, and $\Omega = [0, 2\pi]$. The stochastic parameter ξ takes its values in $\mathscr{E}_2 \times \Omega$. Observe that this problem does not fit into the framework of [8] because ξ is a three-dimensional random variable and Y, the target motion space, is only two-dimensional.

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† Daniel H. Wagner Associates, Paoli, Pennsylvania 19301. This research was supported by the Office of Naval Research under Contract N00014-76-C-0696.

Reasonable, but probably not optimal, plans for maximizing the probability of detection in the fleeing target problem are given in [4]. Although we are not able to find an optimal plan for this problem, we do find necessary and sufficient conditions for such a plan. Unfortunately, it appears difficult to find a plan to satisfy these conditions.

The main results of this paper are to find necessary and sufficient conditions for a plan to maximize the probability of detecting by time T a target which is following a generalized conditionally deterministic motion. In § 2 we describe the target motion mathematically and in § 3, we define the problem of optimal search for a target with generalized conditionally deterministic motion. Section 4 gives sufficient conditions for a search plan to be optimal and § 5 finds necessary conditions for optimality. In § 6, the results of §§ 4 and 5 are specialized to the case of conditionally deterministic motion considered in [8]. The necessary conditions of § 5 are obtained by the use of the methods of Dubovitskii and Milyutin. These methods have also been used to find necessary conditions for the search problem considered in [6].

This paper is very closely related to and draws inspiration from the work of Pursiheimo in [5]. The results in this paper differ from those in [5] mainly by being more general. In particular, a more general parameter space Ω is allowed; the class of functions L over which necessary conditions are found is larger than the class considered in [5]; search effort need not be applied at a constant rate, i.e. the function m defined in § 3 need not be constant; and necessary and sufficient conditions are found when the search densities are bounded as well as unbounded. In addition, no concavity or convexity assumptions are made when proving the necessity results given here. The logic used in proof of the lemma associated with Theorem 2 of [5] can be clarified by the use of Lemma 5.3 of this paper.

2. Description of target motion. Let X and Y be Borel measurable subsets of $\mathscr{E}_n$, and let Ω be a topological space with a σ-finite measure γ defined on the Borel sets of Ω. A target motion function η is given, i.e., η is a Borel function

$$\eta: X\times\Omega\times[0, T]\rightarrow Y,$$

where $T\geqq 0$; note: $T=\infty$ is allowed provided one replaces $[0, T]$ by $[0, \infty)$. There is a stochastic parameter $\boldsymbol{\xi}$ which takes its values in $X\times\Omega$. The distribution of $\boldsymbol{\xi}$ is assumed to be specified by a Borel measurable probability density function p such that for any Borel set $S\subset X\times\Omega$,

$$\Pr\{\boldsymbol{\xi}\in S\}=\int_S p(x, \omega)\gamma(d\omega)\,dx,$$

where dx indicates integration with respect to Lebesgue measure on $\mathscr{E}_n$. The target motion is determined by the value of $\boldsymbol{\xi}$ in the sense that if $\boldsymbol{\xi}=(x, \omega)$, then $\eta(x, \omega, t)$ gives the target's position at all times $t\in[0, T]$. Often we take $X=Y$ and consider x to be the position of the target at time 0. In this case, ω might be the target's velocity.

For $(\omega, t)\in\Omega\times[0, T]$, we shall use the notation $\eta_{\omega t}$ for the transformation $\eta(\cdot\,, \omega, t): X\rightarrow Y$. Let $\eta^i_{\omega t}$ denote the ith component of $\eta_{\omega t}$. For $(\omega, t)\in\Omega\times[0, T]$, we assume that $\eta^i_{\omega t}$ has continuous first partial derivatives at all $x\in X$. Let $\mathbf{J}(x, \omega, t)$

be the absolute value of the Jacobian of the transformation $\eta_{\omega t}$ evaluated at x, i.e.,

$$\mathbf{J}(x, \omega, t) = \left|\det\left(\frac{\partial \eta^i_{\omega t}(x)}{\partial x_j}\right)\right| \quad \text{for } (x, \omega, t) \in X \times \Omega \times [0, T].$$

The target motion function is assumed to have the following properties for almost every $t \in [0, T]$:

(a) $\eta_{\omega t}$ is one-to-one for $(\omega, t) \in \Omega \times [0, T]$,

(b) $\mathbf{J}(x, \omega, t) > 0$ for $(x, \omega, t) \in X \times \Omega \times [0, T]$.

By (a) we may define $\eta^{-1}_{\omega t}$, the inverse of $\eta_{\omega t}$, on range $\eta_{\omega t}$.

3. Statement of search problem. A *search plan density* is a Borel measurable function $\psi : Y \times [0, T] \to [0, \infty)$ which may be thought of as specifying a density for search effort in both time and space. The integral $\int_0^t \psi(y, s)\, ds$ gives the search density applied at a point y by time t and $\int_Y \int_0^t \psi(y, s)\, ds\, dy$, the total search effort expended by time t.

The probability $P_t[\psi]$ of detecting the target by time t with plan ψ is given by

$$P_t[\psi] = \int_{X \times \Omega} p(x, \omega) b\left[\int_0^t \psi(\eta_{\omega s}(x), s)\, ds\right] \gamma(d\omega)\, dx,$$

where b is a detection function. That is, $b[\int_0^t \psi(\eta_{\omega s}(x), s)\, ds]$ gives the probability of detecting the target by time t with plan ψ given that $\boldsymbol{\xi} = (x, \omega)$. It is assumed that b is an increasing function and that $b(0) = 0$. The detection function b could be allowed to depend on x, and all of the results of this paper would remain true upon making the obvious changes. However, for notational simplicity, we consider only the case where b does not depend on x.

We impose the following type of constraint on the search plan ψ. Let $m : [0, T] \to [0, \infty)$ be a Borel function; then we require that allowable search plans satisfy

$$\begin{aligned} &\int_Y \psi(y, t)\, dy \leqq m(t) \quad && \text{for } 0 \leqq t \leqq T, \\ &0 \leqq \psi(y, t) \leqq B && \text{for } (y, t) Y \times [0, T] \end{aligned} \tag{3.1}$$

where B is a positive constant, possibly equal to ∞. Thus, $m(t)$ constrains the global rate at which search effort may be applied at time t, and B constrains the rate at which effort may be applied to small areas. The optimal search problem for this situation is to find a plan ψ^* which maximizes $P_T[\psi]$ over all plans ψ which satisfy (3.1).

4. Sufficient conditions. In order to find necessary and sufficient conditions for an optimal search, we consider the following class of functions. Let L be the set of real-valued Borel functions f defined on $Y \times [0, T]$ such that

$$\|f\| \equiv \int_0^T \operatorname*{ess\,sup}_{y \in Y} |f(y, t)|\, dt < \infty. \tag{4.1}$$

Then (4.1) defines a norm on L (provided we identify functions which are equal almost everywhere), and L is a Banach space in this norm. While this is a rather unusual space, we do note that for $T<\infty$ most optimal search plans that have been found for stationary targets or targets with conditionally deterministic motion are members of this space (see, for example, [8, (4.19)]. This statement would not be true for L_∞, the space of bounded Borel functions on $Y\times[0, T]$. In a manner similar to [8], we let

$$\Psi(B, m, \infty)=\left\{\psi\in L \mid 0\leqq\psi(y, t)\leqq B \text{ for } y\in Y \text{ and } \int_Y \psi(y, t)\,dy \leqq m(t) \text{ for } 0<t<T\right\}.$$

Note, the case where unbounded densities are allowed is obtained by setting $B=\infty$. We say that $\psi^*\in\Psi(B, m, \infty)$ is *t-optimal within* $\Psi(B, m, \infty)$ if

$$P_t[\psi^*]=\max\{P_t[\psi] \mid \psi\in\Psi(B, m, \infty)\}. \tag{4.2}$$

In order to state the sufficient conditions given in Theorem 4.1 below, it is convenient to define the following function. Let b' denote the derivative of b and define

$$D_T(\psi, y, t)=\int_\Omega \frac{p(\eta_{\omega t}^{-1}(y), \omega)}{\mathbf{J}(\eta_{\omega t}^{-1}(y), \omega, t)} b'\left(\int_0^T \psi(\eta_{\omega s}[\eta_{\omega t}^{-1}(y)], s)\,ds\right)\gamma(d\omega) \tag{4.3}$$
$$\text{for } \psi\in\Psi(B, m, \infty),\quad y\in Y,\quad \text{and } 0\leqq t\leqq T.$$

If, for some ω, the point y is not a member of the range of $\eta_{\omega t}$, then we understand the integrand to be 0 for that ω. Let ℓ_r denote r-dimensional Lebesgue measure. We shall use the abbreviation ℓ_r-a.e. to mean almost every in r-dimensional Lebesgue measure.

THEOREM 4.1. *Suppose that b is concave and that for some finite constant* κ, *the derivative,* b', *of b satisfies* $0\leqq b'(z)\leqq\kappa$ *for* $z\geqq 0$. *Let B be a positive extended-real number. If there is a* $\psi^*\in\Psi(B, m, \infty)$ *such that* $\int_X \psi^*(y, t)\,dy=m(t)$ *for* $t\in[0, T]$, *and a Borel function* $\lambda:[0, T]\to[0, \infty)$ *such that for* ℓ_{n+1}*-a.e.* $(y, t)\in Y\times[0, t]$

$$\begin{aligned} D_T(\psi^*, y, t)&\geqq\lambda(t) \quad \text{for } \psi^*(y, t)=B,\\ D_T(\psi^*, y, t)&=\lambda(t) \quad \text{for } 0<\psi^*(y, t)<B,\\ D_T(\psi^*, y, t)&\leqq\lambda(t) \quad \text{for } \psi^*(y, t)=0,\end{aligned} \tag{4.4}$$

then ψ^* *is T-optimal within* $\Psi(B, m, \infty)$.

Proof. Extend the definition of b so that

$$b(z)=b(0)+zb'(0) \quad \text{for } z<0. \tag{4.5}$$

Then $P_T[f]$ is defined for all $f\in L$.

In order to prove the sufficiency of (4.4), we make use of $P'_T[\psi^*, h]$, the derivative of P_T at ψ^* in the direction of h. Recall that

$$P'_T[\psi^*, h]=\lim_{\varepsilon\to 0+}\frac{1}{\varepsilon}(P_T[\psi^*+\varepsilon h]-P_T[\psi^*]) \quad \text{for } h\in L$$

if the limit exists. For $h \in L$,

$$P'_T[\psi^*, h] = \lim_{\varepsilon \to 0+} \int_{X\times\Omega} p(x,\omega)\frac{1}{\varepsilon}\left\{ b\left(\int_0^T [\psi^*(\eta_{\omega s}(x), s) + \varepsilon h(\eta_{\omega s}(x), s)]\, ds\right) - b\left(\int_0^T \psi^*(\eta_{\omega s}(x), s)\, ds\right)\right\}\gamma(d\omega)\, dx.$$

Since b' is bounded by κ, the absolute value of the integrand is bounded by $p(x,\omega)\kappa\|h\|$, and we may apply the dominated convergence theorem to obtain

$$P'_T[\psi^*, h] = \int_{X\times\Omega} p(x,\omega) b'\left(\int_0^T \psi^*(\eta_{\omega s}(x), s)\, ds\right)\int_0^T h(\eta_{\omega t}(x), t)\, dt\, \gamma(d\omega)\, dx \quad \text{for } h \in L.$$

By applying Fubini's theorem and making the change of variable $y = \eta_{\omega t}(x)$, we may write

$$P'_T[\psi^*, h] = \int_0^T \int_Y \int_\Omega \frac{p(\eta_{\omega t}^{-1}(y), \omega)}{\mathbf{J}(\eta_{\omega t}^{-1}(y), \omega, t)} b'\left(\int_0^T \psi^*(\eta_{\omega s}[\eta_{\omega t}^{-1}(y)], s)\, ds\right)\gamma(d\omega) \cdot h(y,t)\, dy\, dt \tag{4.6}$$
$$= \int_0^T \int_Y D_T(\psi^*, y, t) h(y,t)\, dy\, dt.$$

Since b is concave, P_T is a concave functional on L. We show that ψ^* is T-optimal in $\Psi(B, m. \infty)$ by supposing the contrary. That is, suppose there is $\psi \in \Psi(B, m, \infty)$ such that $P_T[\psi] > P_T[\psi^*]$. Then consider the derivative P_T at ψ^* in the direction $h = \psi - \psi^*$. For $0 \leqq \varepsilon \leqq 1$,

$$P_T[\psi^* + \varepsilon h] - P_T[\psi^*] = P_T[(1-\varepsilon)\psi^* + \varepsilon\psi] - P_T[\psi^*]$$
$$\geqq (1-\varepsilon)P_T[\psi^*] + \varepsilon P_T[\psi] - P_T[\psi^*] = \varepsilon(P_T[\psi] - P_T[\psi^*]).$$

Therefore

$$P'_T[\psi^*, h] \geqq P_T[\psi] - P_T[\psi^*] > 0. \tag{4.7}$$

Let $A_1 = \{(y,t) | \psi^*(y,t) = B\}$, $A_2 = \{(y,t) | 0 < \psi^*(y,t) < B\}$ and $A_3 = \{(y,t) | \psi^*(y,t) = 0\}$. Then from (4.4), (4.6), and (4.7), we have

$$0 < P'_T[\psi^*, h] = \int_0^T \int_Y D_T(\psi^*, y, t)[\psi(y,t) - \psi^*(y,t)]\, dy\, dt$$
$$= \sum_{i=1}^3 \int_0^T \int_{A_i} D_T(\psi^*, y, t)[\psi(y,t) - \psi^*(y,t)]\, dy\, dt \tag{4.8}$$
$$\leqq \int_0^T \lambda(t) \int_Y [\psi(y,s) - \psi^*(y,t)]\, dy\, dt \leqq 0.$$

The last inequality follows from the fact that

$$\int_Y \psi(y,t)\,dy \leqq m(t) = \int_Y \psi^*(y,t)\,dy \quad \text{for } 0 \leqq t \leqq T.$$

The contradiction in (4.8) completes the proof of the theorem.

5. Necessary conditions for optimality. In order to find necessary conditions for T-optimality, we use Theorem 6.1 of [2], which is reproduced as Theorem 5.1 below. For the convenience of the reader, some of the terminology used in this theorem is defined below.

Theorem 5.1 gives necessary conditions for the achievement of a constrained minimum of a functional F defined on a locally convex topological vector space E. The contraints are expressed by specifying subsets $Q_i \subset E$ for $i = 1, \cdots, k+1$ and requiring that $f^* \in Q = \bigcap_{i=1}^{k+1} Q_i$ be such that

$$F(f^*) = \min \{F(f) | f \in Q\}.$$

The sets Q_i, $i = 1, \cdots, k$, must contain interior points while Q_{k+1} must have none. Typically, the sets Q_i, $i = 1, \cdots, k$, represent inequality constraints while Q_{k+1} represents an equality constraint. For the purpose of retaining motivation, the sets Q_i, $i = 1, \cdots, k$, are called *inequality sets* while Q_{k+1} is called an *equality set*. Note: there is only one equality set.

A set K in a vector space is called a *cone* with apex at 0 if $h \in K$ implies $\lambda h \in K$ for all $\lambda > 0$. Let E^0 be the set of continuous linear functionals defined on E. Let $K \subset E$ be a cone with apex at 0. The dual cone $\bar{K}$ is defined by

$$\bar{K} = \{g \in E^0 | g(h) \geqq 0 \text{ for } h \in K\}.$$

The vector $h \in E$ is a direction of decrease of F at $f_0 \in E$ if there exists a neighborhood U_h of h and a number $\alpha < 0$ such that for some $\varepsilon_0 > 0$,

$$F(f_0 + \varepsilon\tilde{h}) \leqq F(f_0) + \varepsilon\alpha \quad \text{for } 0 < \varepsilon < \varepsilon_0 \quad \text{and} \quad \tilde{h} \in U_h.$$

One can show that the directions of decrease generate an open cone with apex at 0. The functional F is *regularly decreasing* at $f_0 \in E$ if the set of its directions of decrease at f_0 is convex.

Let $Q \subset E$. The vector $h \in E$ is a *feasible direction* for Q at $f_0 \in E$ if there exists a neighborhood U_h of h such that for some $\varepsilon_0 > 0$

$$f_0 + \varepsilon\tilde{h} \in Q \quad \text{for } 0 < \varepsilon < \varepsilon_0 \quad \text{and} \quad \tilde{h} \in U_h.$$

The set of feasible directions for Q at f_0 form an open cone with apex at 0. The set Q is *feasible regular* at $f_0 \in E$ if the cone of feasible directions for Q at f_0 is convex.

The vector $h \in E$ is a *tangent direction* to Q at $f_0 \in E$ if there exists $\varepsilon_0 > 0$ and a function $q : (0, \varepsilon_0) \to Q$ such that if

$$r(\varepsilon) = q(\varepsilon) - (f_0 + \varepsilon h),$$

then for any neighborhood U of the zero vector, there exists a δ with $0 \leqq \delta \leqq \varepsilon_0$ such that

$$(1/\varepsilon) r(\varepsilon) \in U \quad \text{for } 0 < \varepsilon < \delta.$$

In a Banach space, this last condition may be replaced by $\|r(\varepsilon)\| = o(\varepsilon)$. One may verify that the tangent directions to Q at f_0 form a cone with apex at 0, but this cone is, in general, neither open nor closed. Note that every feasible direction is also a tangent direction but the converse is not true. The set $Q \subset E$ is *tangent regular* at f_0 if the cone of tangent directions to Q at f_0 is convex. We now state the theorem of Dubovitskii and Milyutin which we will use in proving our results.

THEOREM 5.1. (Dubovitskii–Milyutin). *Let the functional F be defined on a locally convex topological vector space E. Let $Q_1, \cdots, Q_k$ be inequality subsets of E and Q_{k+1} be an equality subset of E. Let F assume a minimum at $f^* \in Q = \bigcap_{i=1}^{k+1} Q_i$. Assume*

1) *F is regularly decreasing at f^* with cone of directions of decrease K_0;*
2) *for $i = 1, \cdots, k$, Q_i is feasible regular at f^* with cone of feasible directions K_i, which is nonempty;*

and

3) *the set Q_{k+1} is tangent regular at f^* with a cone of tangent directions K_{k+1}, which contains at least one nonzero vector.*

Then there exist continuous linear functionals g_i, $i = 0, \cdots, k+1$, not all identically zero such that $g_i \in \bar{K}_i$ = dual cone of K_i for $i = 0, \cdots, k+1$, which satisfy the Euler–Lagrange equation

$$g_0 + g_1 + \cdots + g_{k+1} = 0. \tag{5.1}$$

As before we let ℓ_r denote r-dimensional Lebesgue measure and use the abbreviation ℓ_r-a.e. to mean almost every in r-dimensional Lebesgue measure. The following theorem is the main theorem of this paper.

THEOREM 5.2. *Suppose there is a real number κ such that $0 < b'(z) \leqq \kappa$ for $z \geqq 0$. Let B be a positive extended-real number and m a Borel function such that m: $[0, T] \to [0, \infty)$ and $0 < m(t) < B\ell_n(Y)$ for $t \in [0, T]$. If ψ^* is T-optimal within $\Psi(B, m, \infty)$, then there exists $\lambda : [0, T] \to [0, \infty)$ such that for ℓ_{n+1}-a.e. $(y, t) \in Y \times [0, T]$*

$$\begin{aligned} D_T(\psi^*, y, t) &\geqq \lambda(t) \quad \text{if } \psi^*(y, t) = B, \\ D_T(\psi^*, y, t) &= \lambda(t) \quad \text{if } 0 < \psi^*(y, t) < B, \\ D_T(\psi^*, y, t) &\leqq \lambda(t) \quad \text{if } \psi^*(y, t) = 0. \end{aligned} \tag{5.2}$$

Proof. In the terminology of Theorem 5.1, we let $k = 1$;

$$\begin{aligned} &E = L, \qquad Q_1 = L, \\ &Q_2 = \{f \in L \mid 0 \leqq f(y, t) \leqq B \text{ for } (y, y) \in Y \times [0, T] \\ &\qquad \text{and } \textstyle\int_Y f(y, t)\, dy \leqq m(t) \text{ for } t \in [0, T]\} \\ &F(f) = -P_T(f) \quad \text{for } f \in L. \end{aligned}$$

Note the definition of b is extended as given in (4.5). The set Q_2 is the equality set, i.e., it has no interior points in the norm topology of L.

We find K_0 by use of Theorem 7.3 of [2]. To do this, we first show that F satisfies a Lipschitz condition. Since $|b'(z)| \leqq \kappa$ for $z \geqq 0$,

$$|F(f_2)-F(f_1)| \leqq \int_{X\times\Omega} p(x,\omega)\kappa \left| \int_0^T [f_1(\eta_{\omega s}(x),s)-f_2(\eta_{\omega s}(x),s)]\,ds \right| \gamma(d\omega)\,dx$$

$$\leqq \int_{X\times\Omega} p(x,\omega)\kappa \int_0^T \operatorname*{ess\,sup}_{y\in Y} |f_1(y,s)-f_2(y,s)|\,ds\gamma(d\omega)\,dx$$

$$\leqq \kappa \|f_1-f_2\|.$$

Let ψ^* be T-optimal within $\Psi(B,m,\infty)$ and let $F'(\psi^*,h)$ be the derivative of F at ψ^* in the direction of h. Since $F=-P_T$, we have by (4.6) that

$$F'(\psi^*,h) = -\int_0^T \int_Y D_T(\psi^*,y,t)h(y,t)\,dy\,dt. \tag{5.3}$$

For convenience, let

$$\delta(y,t) = D_T(\psi^*,y,t) \quad \text{for } y\in Y, \quad 0\leqq t\leqq T. \tag{5.4}$$

Observe that δ is nonnegative, and that by making the change of variable $x=\eta_{\omega t}^{-1}(y)$ and using (4.3), one can verify that there is a finite constant M such that

$$\int_Y \delta(y,t)\,dy = M \quad \text{for } 0\leqq t\leqq T.$$

It now follows that $F'(\psi^*,h)$ is a bounded linear operator on L, i.e.,

$$|F'(\psi^*,h)| = \left| \int_0^T \int_Y \delta(y,t)h(y,t)\,dy\,dt \right|$$

$$\leqq M \int_0^T \operatorname*{ess\,sup}_{y\in Y} |h(y,t)|\,dt = M\|h\| \quad \text{for } h\in L.$$

By Theorem 7.3 of [2], F is regularly decreasing at ψ^* and the cone of directions of decrease is

$$K_0 = \{h\in L \mid F'(\psi^*,h)<0\}.$$

Let L^0 be the dual space of L. By Theorem 10.2 of [2], the dual cone

$$\bar{K}_0 = \{g\in L^0 \mid \text{for some } \alpha\geqq 0,\ g(h) = -\alpha F'(\psi^*,h) \text{ for } h\in L\}.$$

Obviously, the cone $K_1 = L$ and $\bar{K}_1 = \{g_1\}$ where $g_1(h)=0$ for $h\in L$. Instead of identifying K_2 and $\bar{K}_2$, we shall show only that

$$g(\psi)\geqq g(\psi^*) \quad \text{for } g\in \bar{K}_2 \quad \text{and} \quad \psi\in Q_2. \tag{5.5}$$

This will be sufficient for us to show that conditions (5.2) are satisfied.

In order to prove (5.5), we fix an arbitrary $\psi\in Q_2$ and let

$$h = \psi-\psi^*.$$

Then the convexity of Q_2 implies $h\in K_2$. To see this, let

$$q(\varepsilon) = (1-\varepsilon)\psi^* + \varepsilon\psi \quad \text{for } 0<\varepsilon<1.$$

Then $q(\varepsilon)\in Q_2$ and $r(\varepsilon)=q(\varepsilon)-(\psi^*+\varepsilon h)=0$ for $0<\varepsilon<1$. Thus, $h\in K_2$. Let $g\in \bar{K}_2$. By definition, $g(f)\geqq 0$ for $f\in K_2$ and we have $g(h)=g(\psi-\psi^*)\geqq 0$. This proves (5.5) and shows that K_2 contains a nonzero vector.

One may show that the convexity of Q_2 implies that K_2 is convex and thus that Q_2 is tangent regular at ψ^*. By Theorem 5.1, there exist $g_i\in\bar{K}_i$ for $i=0, 1, 2$, which satisfy $g_0+g_1+g_2=0$. Since $g_1=0$, it follows that $g_2=-g_0$; i.e., for some $\alpha>0$,

$$\begin{aligned} g_2(h) &= -g_0(h) \\ &= -\alpha\int_0^T\int_Y \delta(y,t)h(y,t)\,dy\,dt \quad \text{for } h\in L. \end{aligned} \tag{5.6}$$

In order to complete our proof, we show that the following claim is true.

CLAIM. *For ℓ_1-a.e. $t\in[0,T]$*

$$\gamma(t)\equiv\int_Y \delta(y,t)\psi^*(y,t)\,dy \geqq \int_Y \delta(y,t)\beta(y)\,dy \tag{5.7}$$

for all real-valued measurable functions $\beta: Y\to[0,B]$ such that $\int_Y\beta(y)\,dy\leqq m(t)$.

Suppose the claim fails. That is, there is a Borel set $R\subset[0,T]$ such that $\ell_1(R)>0$ and a collection of functions $\{\beta_t : t\in R\}$ such that β_t violates (5.7) for $t\in R$. By Lemma 5.3, proved below, there is a real-valued Borel measurable function $\hat\beta$ defined on $Y\times R$ such that

(5.8)
(i) $\hat\beta(\cdot,t)$ is continuous with compact support, $\hat\beta(y,t)\in[0,B]$ for $y\in Y$,
(ii) $\int_Y\hat\beta(y,t)\,dy\leqq m(t)$,
and (iii) $\int_Y\delta(y,t)\hat\beta(y,t)\,dy>\gamma(t)$ for ℓ_1-a.e. $t\in R$.

One can show that the function h defined by $h(t)=\sup_{y\in Y}\hat\beta(y,t)$ for $t\in R$ is ℓ_1-measurable. Thus by a corollary to Lusin's theorem, it is ℓ_1-a.e. equal to a Borel function, and we can choose a Borel set $S\subset R$ and a number $\tilde B<\infty$ such that $0<\ell_1(S)<\infty$ and

$$\sup_{y\in Y}\hat\beta(y,t)\leqq\tilde B \quad \text{for } t\in S.$$

Now let

$$\psi(y,t)=\begin{cases}\hat\beta(y,t) & \text{for } (y,t)\in Y\times S,\\ \psi^*(y,t) & \text{for } (y,t)\in Y\times[0,T]\backslash S.\end{cases}$$

Then $\psi\in Q_2$ and by (5.6) and (5.8), $g_2(\psi)<g_2(\psi^*)$, contradicting (5.5). Thus, the claim must hold.

Since $0<m(t)<B\ell_n(Y)$, clearly $0<\int_Y\psi^*(y,t)\,dy<B\ell_n(Y)$, and since the claim holds, we may apply a standard Lagrange multiplier result (e.g., [7, Thm. B, 2.4]) to guarantee the existence, for ℓ_1-a.e. $t\in[0,T]$, of $\lambda(t)\geqq 0$ such that for ℓ_n-a.e. $y\in Y$

$$\delta(y,t)\psi^*(y,t)-\lambda(t)\psi^*(y,t)\geqq\delta(y,t)z-\lambda(t)z \quad \text{for finite } z\in[0,B]. \tag{5.9}$$

Since $\delta(y,t)=D_T(\psi^*,y,t)$, it is clear that (5.9) can hold only if (5.2) holds. This proves the theorem.

LEMMA 5.3. *Suppose Y is a Borel subset of Euclidean n-space. Let m, δ, and γ be Borel functions such that $m:[0, T]\to(0, \infty)$, $\delta: Y\times[0, T]\to[0, \infty)$ satisfies $\int_Y \delta(y, t)\,dy < \infty$ for $t\in[0, T]$, and $\gamma:[0, T]\to[0, \infty)$. Let $R\subset[0, T]$ be a Borel set such that $\ell_1(R)>0$. Suppose that $B\in(0, \infty]$ and that for each $t\in R$, there exists a real-valued Borel measurable function $\beta_t: Y\to[0, B]$ such that*

$$(5.10)\qquad \begin{aligned} &\text{(i)} \int_Y \beta_t(y)\,dy \leqq m(t),\\ \text{and } &\text{(ii)} \int_Y \delta(y, t)\beta_t(y)\,dy > \gamma(t).\end{aligned}$$

Then there exists a real-valued Borel measurable function $\hat\beta$ defined on $Y\times R$ such that for ℓ_1-a.e. $t\in R$, $\hat\beta(\cdot\,, t)$ is a continuous function into $[0, B]$ with compact support,

$$(5.11)\qquad \int_Y \delta(y, t)\hat\beta(y, t)\,dy > \gamma(t) \quad \textit{and} \quad \int_Y \hat\beta(y, t)\,dy \leqq m(t).$$

Proof. We first show that we may take β_t to be a continuous function with compact support for $t\in R$. Fix $t\in R$. Let $S_m = \{y: \beta_t(y)\leqq m\}$. Since

$$\int_Y \delta(y, t)\beta_t(y)\,dy = \lim_{m\to\infty}\int_{S_m} \delta(y, t)\beta_t(y)\,dy$$

we may choose M such that

$$\int_{S_M} \delta(y, t)\beta_t(y)\,dy > \gamma(t).$$

It is now clear that β_t may be replaced by a bounded function $\beta_t^{\#}$ which will satisfy (i) and (ii) when substituted for β_t in (5.10).

Let U_m be the closed ball of radius m centered at the origin of Euclidean n-space. By a similar argument it is clear that we may choose $\beta_t^{\#}$ to have compact support and to satisfy strict inequalities in both (i) and (ii). Thus

$$a \equiv \int_Y \delta(y, t)\beta_t^{\#}(y)\,dy - \gamma(t) > 0, \qquad b \equiv m(t) - \int_Y \beta_t^{\#}(y)\,dy > 0.$$

Choose r so that $\{y: \beta_t^{\#}(y)>0\}\subset U_r$, and let $Z = Y\cap U_r$.

By Lusin's theorem, there is a sequence V_i, $i = 1, 2, \cdots$, of compact subsets of Z and continuous functions α_i defined on V_i for $i = 1, 2, \cdots$, such that $\lim_{i\to\infty} \ell_n(Z\setminus V_i) = 0$ and $\alpha_i = \beta_t^{\#}$ on V_i for $i = 1, 2, \cdots$. By the Tietze extension theorem, each α_i may be extended to a continuous function defined on all of Z in such a manner that $\alpha_i(y)\geqq 0$ for $y\in Z$ and $\sup_{y\in Z}\alpha_i(y)\leqq \Gamma \equiv \sup_{y\in Y}\beta_t^{\#}(y)<\infty$ for $i = 1, 2, \cdots$. For each i we may extend α_i to all of Y as follows. For n-vectors y, let $|y|$ be the distance of y from the origin. For $y\in Y$ define

$$\hat\alpha_i(y) = \begin{cases} \alpha_i(y) & \text{for } |y|\leqq r,\\ (ir+1-i|y|)\alpha\left(\dfrac{r}{|y|}y\right) & \text{for } r<|y|\leqq r+1/i,\\ 0 & \text{for } |y|>r+1/i.\end{cases}$$

Then $\hat{\alpha}_i$ is continuous, has compact support, agrees with $\beta_t^{\#}$ on V_i and on $Y\backslash U_{r+1/i}$, and satisfies $\sup_{y\in Y}\hat{\alpha}_i(y)\leqq\Gamma$.

For $i=1,2,\cdots$, let $W_i=(Z\backslash V_i)\cup\{Y\cap(U_{r+1/i}\backslash U_r)\}$. Then

$$\int_Y \delta(y,t)|\hat{\alpha}_i(y)-\beta_t^{\#}(y)|\,dy\leqq\Gamma\int_{W_i}\delta(y,t)\,dy$$

and

$$\int_Y |\hat{\alpha}_i(y)-\beta_t^{\#}(y)|\,dy\leqq\Gamma\ell_n(W_i)\quad\text{for } i=1,2,\cdots.$$

Since $\delta(\cdot\,,t)$ is finitely integrable, it defines a measure on Y which is absolutely continuous with respect to ℓ_n. Because $\lim_{i\to\infty}\ell_n(W_i)=0$ we may choose N so that

$$\int_Y \delta(y,t)|\hat{\alpha}_N(y)-\beta_t^{\#}(y)|\,dy<a/2$$

and

$$\int_Y |\hat{\alpha}_N(y)-\beta_t^{\#}(y)|\,dy<b/2.$$

Thus, we may replace β_t by the function $\hat{\alpha}_N$ in (5.10) and the relations (i)–(ii) continue to hold. For the remainder of the proof, we shall simply assume that β_t is a continuous function with compact support for $t\in R$.

Let $C(Y)$ be the metric space of continuous functions on Y having compact support under the supremum metric. Then $C(Y)$ is separable but not complete. Embed $C(Y)$ in its completion $\hat{C}(Y)$. For each $m=1,2,\cdots$ the subset of functions in $C(Y)$ with support in U_m is a closed subset of $\hat{C}(Y)$. Thus $C(Y)$ is a countable union of closed subsets, and hence is a Borel subset of $\hat{C}(Y)$. This means that $C(Y)$ is standard in the terminology of Aumann [1]. This fact will be used later in order to invoke Aumann's measurable choice theorem. For $t\in R$, define

$$\mathcal{F}(t)=C(Y)\cap\Big\{\alpha\,|\,0\leqq\alpha(y)<B\text{ for } y\in Y,\int_X\delta(y,t)\alpha(y)\,dy>\gamma(t)\quad\text{and}\int_X\alpha(y)\,dy\leqq m(t)\Big\},$$

and define

$$\varphi(t,\alpha)=\int_Y\delta(y,t)\alpha(y)\,dy\quad\text{for }\alpha\in C(Y).$$

Then $\varphi(t,\cdot)$ is a continuous function on $C(Y)$ for $t\in R$ and $\varphi(\cdot\,,\alpha)$ is Lebesgue measurable for $\alpha\in C(Y)$. Let $\mathcal{B}$ be the σ-algebra of Borel subsets of $C(Y)$ and $\mathcal{M}$ the σ-algebra of Borel subsets of R. By Theorem 6.1 of [3], φ is $\mathcal{M}\times\mathcal{B}$-measurable. Let $\xi(\alpha)=\int_Y\alpha(y)\,dy$ for $\alpha\in C(Y)$. One can verify that ξ is a Borel function by observing that for a fixed real number r, $\{\alpha\,|\,\xi(\alpha)\leqq r\}$ is the countable union of the following closed subsets of $C(Y)$:

$$\{\alpha : \xi(\alpha) \leqq r \text{ and } \alpha \text{ has its support in } U_m\}, \qquad m = 1, 2, \cdots.$$

It is now clear that graph $\mathcal{F} = \{(t, \alpha) : \alpha \in \mathcal{F}(t)\} \in \mathcal{M} \times \mathcal{B}$. Since $\mathcal{F}(t)$ is nonempty for $t \in R$, we have by Aumann's measurable choice theorem [1] that there is a Borel function h defined on R such that $h(t) \in \mathcal{F}(t)$ for ℓ_1-a.e. $t \in R$.

Let

$$\hat{\beta}(y, t) = h(t)(y) \quad \text{for } (y, t) \in Y \times R.$$

To show that $\hat{\beta}$ is a Borel function, we invoke Lusin's theorem to obtain a countable collection $\{V_1, V_2, \cdots\}$ of disjoint Borel subsets of R such that $\ell_1(R \setminus \bigcup_{i=1}^{\infty} V_i) = 0$ and for each i, h restricted to V_i is continuous. Let $\hat{\beta}_i$ be β restricted to $Y \times V_i$. Then for any $y \in Y$, $\hat{\beta}_i(y, \cdot)$ is continuous. Since for any $t \in V_i$, $\hat{\beta}_i(\cdot, t)$ is also continuous, we have by Theorem 6.1 of [3] that $\hat{\beta}_i$ is Borel measurable on $V_i \times Y$. Thus $\hat{\beta}$ is equal ℓ_{n+1}-a.e. to a Borel function.

Since $h(t) \in \mathcal{F}(t)$ for ℓ_1 a.e. $t \in R$, it follows that for ℓ_1 a.e. $t \in R$, $\hat{\beta}(\cdot, t)$ is continuous and has compact support, $0 \leqq \hat{\beta}(y, t) \leqq B$ for $y \in Y$ and $\hat{\beta}(\cdot, t)$ satisfies (5.11). This proves the lemma.

6. The special case of conditionally deterministic motion. When we are dealing with the type of conditionally deterministic motion considered in [8], it is possible to simplify the conditions obtained in Theorems 4.1 and 5.2. For this case the stochastic parameter $\boldsymbol{\xi}$ takes its values in X rather than $X \times \Omega$. Or, in the notation of this paper, Ω becomes a singleton set $\{\omega_0\}$ and γ places probability one upon that singleton. In this case, the function D_T given in (4.3) becomes

$$D_T(\psi, y, t) = p(\eta_t^{-1}(y)) b'\left(\int_0^T \psi(\eta_s[\eta_t^{-1}(y)], s)\, ds\right) \Big/ \mathbf{J}(\eta_t^{-1}(y), t). \tag{6.1}$$

In the above equation, the dependence on ω_0 is trivial and is, therefore, not indicated.

Using (6.1), we may write (4.4) and (5.2) as

$$\begin{aligned} p(\eta_t^{-1}(y)) b'\left(\int_0^T \psi^*(\eta_s[\eta_t^{-1}(y)], s)\, ds\right) &\geqq \lambda(t) \mathbf{J}(\eta_t^{-1}(y), t) \quad \text{if } \psi^*(y, t) = B, \\ &= \lambda(t) \mathbf{J}(\eta_t^{-1}(y), t) \quad \text{if } 0 < \psi^*(y, t) < B, \\ &\leqq \lambda(t) \mathbf{J}(\eta_t^{-1}(y), t) \quad \text{if } \psi^*(y, t) = 0. \end{aligned} \tag{6.2}$$

Letting $x = \eta_t^{-1}(y)$, (6.2) becomes

$$\begin{aligned} p(x) b'\left(\int_0^T \psi^*(\eta_s(x), s)\, ds\right) &\geqq \lambda(t) \mathbf{J}(x, t) \quad \text{if } \psi^*(\eta_t(x), t) = B, \\ &= \lambda(t) \mathbf{J}(x, t) \quad \text{if } 0 < \psi^*(\eta_t(x), t) < B, \\ &\leqq \lambda(t) \mathbf{J}(x, t) \quad \text{if } \psi^*(\eta_t(x), t) = 0. \end{aligned} \tag{6.3}$$

The above conditions are easily seen to be a generalization of those given in Theorem 8.4.1 of [7].

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