TOTAL OPTIMALITY OF INCREMENTALLY OPTIMAL ALLOCATIONS*

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ABSTRACT

This paper considers the problem of finding optimal solutions to a class of separable constrained extremal problems involving nonlinear functionals. The results are proved for rather general situations, but they may be easily stated for the case of search for a stationary object whose a priori location distribution is given by a density function on R, a subset of Euclidean n-space. The functional to be optimized in this case is the probability of detection and the constraint is on the amount of effort to be used.

Suppose that a search of the above type is conducted in such a manner as to produce the maximum increase in probability of detection for each increment of effort added to the search. Then under very weak assumptions, it is proven that this search will produce an optimal allocation of the total effort involved. Under some additional assumptions, it is shown that any amount of search effort may be allocated in an optimal fashion.

1. INTRODUCTION

In this paper we consider the relationship between incrementally optimal allocations and totally optimal allocations. Motivation for studying this relationship arises naturally in planning a search for a lost object. Suppose that the search planner is given authorization to search for a fixed time interval, and he conducts the search to produce the maximum probability of detection at the end of the interval. If the search fails to detect the lost object within the allotted time, the planner may be given authorization to continue searching for an additional time increment. In this case the planner may allocate the additional search effort to maximize the probability of detection in the given increment. Having done this, one may ask whether the search could have produced a higher detection probability if it were known in advance that both the initial time interval and the added increment were available.

In mathematical terms the search problem is to allocate optimally a given amount of effort in order to detect a stationary object, the target, located in Euclidean n-space, R. There is a function f which gives the probability density of the target's location. Suppose T is the amount of effort available for the search. Then the search planner seeks a function $q^*:R\rightarrow [0,\infty)$ such that $\int_R q^*(x)dx \le T$ and

$$(1.1) \qquad \int_R b(x,q^*(x))f(x)dx = \max\left\{\int_R b(x,q(x))f(x)dx : q\ge 0, \int_R q(x)dx \le T\right\}\cdot$$

The function $b(x,\cdot)$ is the *local effectiveness function* at x. That is, $b(x,y)$ gives the conditional probability of detecting the target given it is located at x and the effort density is y at x. The integral on the left of (1.1) gives the probability of detecting the target when using allocation q^*. The function q^* is called an optimal allocation. This problem has an obvious analog when R is replaced by a countable set of locations or boxes.

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For the case where $b(x, y)=1-e^{-y}$ for $x\epsilon R$ and $y\geq 0$, Koopman [4, p. 617] made the following observation. Suppose one allocates T_1 amount of effort in an optimal fashion, but fails to detect the target. An increment T_2 of effort then becomes available. If one allocates this additional effort in an incrementally optimal manner (i.e., optimal considering the previous allocation of T_1 amount of effort), then one obtains an optimal allocation of T_1+T_2 effort. That is, two incrementally optimal allocations produce a totally optimal allocation.

In [2] an incomplete attempt was made to show that incrementally optimal allocations produce totally optimal allocations provided that $\partial b(x, y)/\partial y$ is a positive monotonic nonincreasing function of y for $x\epsilon R$. In section 2 of this paper we show that for any Borel measurable local effectiveness function, incrementally optimal allocations are totally optimal whenever the target's probability distribution is given by a density function as in (1.1). In the case where the search space is countable, we prove that concavity of the local effectiveness function guarantees that incrementally optimal allocations are totally optimal. In addition, it is shown by counterexample that this property need not hold for countable search spaces if the local effectiveness function is not concave.

A search plan is called uniformly optimal if it maximizes the probability of detection at each instant during the search. In section 3, we show the existence of uniformly optimal search plans under additional hypotheses which are given there.

Our results hold in a more general situation than that of search theory. Thus, we introduce the following framework which is substantially the same as that in [6], one difference being that we deal only with Borel functions. Let R be a Borel subset of Euclidean n-space. We fix Borel functions L and U with $L\leq U$ which are defined on R. The functions L and U may take infinite values.

Define $\Omega=\{(x, y): x\epsilon R,\ |y|<\infty \text{ and } L(x)\leq y\leq U(x)\}$. We fix a real-valued Borel function e defined on Ω and the family Ξ of a.e. (with respect to Lebesgue measure) real-valued Borel functions q defined on R such that $L\leq q\leq U$. For $q\epsilon\Xi$ we understand $e(\cdot, q(\cdot))$ to be a function from R to the reals. Define

$$\Phi=\Xi\cap\{q: e(\cdot, q(\cdot)) \text{ and } q \text{ are integrable}\},$$

and let

$$E(q)=\int_R e(x, q(x))dx \text{ and } C(q)=\int_R q(x)dx \qquad \text{for } q\epsilon\Phi$$

All integration is Lebesgue integration. A $q^*\epsilon\Phi$ is said to be *optimal* if

$$E(q^*)=\max\{E(q): q\epsilon\Phi \text{ and } C(q)=C(q^*)\}.$$

In the case where $L(x)=0$, $U(x)=\infty$ for $x\epsilon R$ and $e(x, y)=f(x)b(x, y)$ for $(x, y)\epsilon\Omega$, $E(q)$ becomes the probability of detecting the target with allocation q and $C(q)$ becomes the amount of effort required by q. Then an optimal q^* maximizes the probability of detection which can be obtained with effort $C(q^*)$.

A function f defined on the real line is said to be *increasing* if $y\geq x$ implies $f(y)\geq f(x)$. A function f is said to be *concave* if for all x, y in the domain of f, $f(\alpha x+(1-\alpha)y))\geq\alpha f(x)+(1-\alpha)f(y)$ for $0\leq\alpha\leq 1$.

2. INCREMENTAL OPTIMIZATION

For $i=1, 2, \ldots$, let $q_i\epsilon\Phi$ be such that $q_1\leq q_2\leq\ldots$. Let $q_0=L$. If

$$E(q_i) = \max \{E(q) : q \geqslant q_{i-1}, q \epsilon \Phi \text{ and } C(q) = C(q_i)\} \qquad \text{for } i = 1, 2, \ldots,$$

then we say that $(q_1, q_2, \ldots)$ is an *incrementally optimal* sequence. If q_i satisfies

$$E(q_i) = \max \{E(q) : q \epsilon \Phi \text{ and } C(q) = C(q_i)\} \qquad \text{for } i = 1, 2, \ldots,$$

then $(q_1, q_2, \ldots)$ is said to be a *totally optimal* sequence. Define

$$\ell(x, y, \lambda) = e(x, y) - \lambda y \qquad \text{for } -\infty < \lambda < \infty, \text{ and } (x, y) \epsilon \Omega.$$

The function ℓ is called a pointwise Lagrangian in [6] and λ is a Lagrange multiplier.

THEOREM 2.1: Let $(q_1, q_2, \ldots)$ be an incrementally optimal sequence such that for $i = 1, 2, \ldots$ $|E(q_i)| < \infty$ and $C(q_i)$ is in the interior of the range of C. Then $(q_1, q_2 \ldots)$ is a totally optimal sequence.

PROOF: By the definition of incremental optimality, q_1 is optimal. Thus, by Corollary 5.2 of [6], there exists a real number λ_1 such that for a.e. $x \epsilon R$

$$\ell(x, q_1(x), \lambda_1) \geqslant \ell(x, y, \lambda_1) \qquad \text{for } |y| < \infty \text{ and } L(x) \leqslant y \leqslant U(x). \tag{2.1}$$

In other words a necessary condition for q_1 to be optimal is that it maximize a pointwise Lagrangian for some multiplier λ_1. Similarly, the incrementally optimal nature of q_2 implies the existence of a real number λ_2 such that for a.e. $x \epsilon R$

$$\ell(x, q_2(x), \lambda_2) \geqslant \ell(x, y, \lambda_2) \qquad \text{for } |y| < \infty \text{ and } q_1(x) \leqslant y \leqslant U(x). \tag{2.2}$$

In order to prove that q_2 is optimal it is sufficient to find a real number λ such that for a.e. $x \epsilon R$

$$\ell(x, q_2(x), \lambda) \geqslant \ell(x, y, \lambda) \qquad \text{for } |y| < \infty \text{ and } L(x) \leqslant y \leqslant U(x). \tag{2.3}$$

The sufficiency of (2.3) follows from a well known result concerning Lagrange multipliers (see, for example [3], [8] or Theorem 2.1 of [6]).

By (2.1) and (2.2)

$$\lambda_2(q_2(x) - q_1(x)) \leqslant e(x, q_2(x)) - e(x, q_1(x)) \leqslant \lambda_1(q_2(x) - q_1(x)) \qquad \text{for a.e. } x \epsilon R. \tag{2.4}$$

Recall that $q_2 \geqslant q_1$. If $q_2(x) = q_1(x)$ for a.e. $x \epsilon R$, then (2.3) holds for $\lambda = \lambda_1$. If $q_2(x) > q_1(x)$ for x in a set of positive measure, then (2.4) implies that $\lambda_2 \leqslant \lambda_1$. In this case for a.e. $x \epsilon R$ and y such that $|y| < \infty$ and $L(x) \leqslant y \leqslant q_1(x)$, we have

$$0 \leqslant e(x, q_1(x)) - e(x, y) - \lambda_1(q_1(x) - y) \leqslant e(x, q_1(x)) - e(x, y) - \lambda_2(q_1(x) - y).$$

That is for a.e. $x \epsilon R$,

$$\ell(x, y, \lambda_2) \leqslant \ell(x, q_1(x), \lambda_2) \leqslant \ell(x, q_2(x), \lambda_2) \qquad \text{for } |y| < \infty, L(x) \leqslant y \leqslant q_1(x). \tag{2.5}$$

Combining (2.5) and (2.2) we obtain (2.3) with $\lambda=\lambda_2$. Thus, q_2 is optimal. By repeating the argument for $q_3, q_4, \ldots$, the theorem is proved.

We now shift our attention to the case where R is a countable set. That is for some countable subset $\mathbf{J}$ of the integers, $R=\{x_j : j\in\mathbf{J}\}$. Let

$$\tilde{E}(q)=\sum_{j\in\mathbf{J}} e(x_j, q(x_j))$$

$$\tilde{C}(q)=\sum_{j\in\mathbf{J}} q(x_j).$$

Carry over the definitions of incrementally and totally optimal sequences in the obvious way. One may use the method of proof given in Theorem 2.1 to show that incrementally optimal sequences are totally optimal for the case where R is countable provided that the existence of a real number λ such that

$$\ell(x_j, q^*(x_j), \lambda)=\sup\{\ell(x_j, y, \lambda) : |y|\leqslant\infty \text{ and } L(x)<y\leqslant U(x)\} \qquad \text{for } j\in\mathbf{J} \tag{2.6}$$

is a necessary condition for q^* to satisfy

$$\tilde{E}(q^*)=\max\{\tilde{E}(q) : q\in\Phi \text{ and } \tilde{C}(q)=\tilde{C}(q^*)\}. \tag{2.7}$$

From Corollary 5.3 and Remark 2.3 of [6] we conclude that if $e(x_j, \cdot)$ is a concave function for $j\in\mathbf{J}$, then (2.6) is necessary for (2.7). Thus, we may state the following theorem.

THEOREM 2.2: If R is countable and $e(x_j, \cdot)$ is concave for $j\in\mathbf{J}$, then any incrementally optimal sequence is totally optimal.

The following example shows that one cannot remove the assumption that $e(x_j, \cdot)$ is concave in Theorem 2.2. The example also shows that (2.6) is not necessary for (2.7) when $e(x_j, \cdot)$ is not concave for $j\in\mathbf{J}$.

EXAMPLE 2.3: Let $R=\{1, 2\}$ be a doubleton set, $L=0$, $U(1)=2$, and $U(2)=\sqrt{3}$. Define

$$e(1, y)=\frac{1}{2}y \quad 0\leqslant y\leqslant 2, \qquad e(2, y)=\begin{cases}\frac{1}{2}y & 0\leqslant y\leqslant 1\\ \frac{1}{4}y^2+\frac{1}{4} & 1<y\leqslant\sqrt{3}.\end{cases}$$

Note that both $e(1, \cdot)$ and $e(2, \cdot)$ are everywhere differentiable. For $0\leqslant T\leqslant 2+\sqrt{3}$, define

$$\eta(1, T)=\begin{cases}0, & 0\leqslant T\leqslant\sqrt{3}\\ T-\sqrt{3}, & \sqrt{3}\leqslant T\leqslant 2+\sqrt{3}\end{cases}$$

$$\eta(2, T)=\begin{cases}T, & 0\leqslant T\leqslant\sqrt{3}\\ \sqrt{3}, & \sqrt{3}<T\leqslant 2+\sqrt{3}.\end{cases}$$

Then $\eta(i, \cdot)$, $i=1, 2$, is increasing, and for each $T\geqslant 0$, $\tilde{C}(\eta(\cdot, T))=T$ and $\tilde{E}(\eta(\cdot, T))$ gives the maximum of $\tilde{E}(q)$ over all nonnegative functions q defined on $\{1, 2\}$ such that $\tilde{C}(q)=T$. Note that for $q^* = \eta(\cdot, 1)$, (2.6) is *not* satisfied for any λ. An example of a function q^* which satisfies (2.7), but for which there is no λ satisfying (2.6) is also given in [8].

One may check that the sequence of allocations (q_1, q_2), where $q_1(1)=1$, $q_1(2)=0$ and $q_2(1)=1$, $q_2(2)=1$, is incrementally optimal. However, $\tilde{E}(q_2)=1$ and $\tilde{C}(q_2)=2$, while

$$\tilde{E}(\eta(\cdot, 2))=2-\tfrac{1}{2}\sqrt{3}>1$$

so that q_2 is not optimal, i.e., (q_1, q_2) is not totally optimal.

In [2, p. 328] it is claimed that (in our notation) the existence of a function η defined on $R\times[0, S]$ such that $\eta(\cdot, T)$ is an optimal allocation of T amount of effort for each $0 \leqslant T \leqslant S$ and $\eta(x, \cdot)$ is increasing for $x \epsilon R$ guarantees that incrementally optimal sequences are totally optimal. Example 2.3 shows that for discrete R this claim does not hold. If in addition to the existence of a function η satisfying the above conditions we have that for each amount of effort there is an almost everywhere *unique* optimal allocation of that effort, then any incrementally optimal sequence is totally optimal. Although not stated as such, this result is proven in [2].

Example 2.3 shows, of course, that optimal allocations need not be unique. Even when E and C are defined as integrals with respect to Lebesgue measure on n-space as is done in section 1, optimal allocations need not be unique. In fact, it is easy to see that if there exists a subset D of R having positive measure such that for $x \epsilon D$ the graph of $e(x, \cdot)$ contains a nondegenerate straight-line segment of slope λ, then there are amounts of effort for which an optimal allocation of that effort is not almost everywhere unique.

REMARK 2.4: Let us return to the search situation described in section 1. That is, $L(x)=0$, $U(x)=\infty$ for $x\epsilon R$, $e(x, y)=f(x)b(x, y)$ for $(x, y)\epsilon\Omega$. Suppose that an optimal allocation q_1 has been performed and that the search has failed to detect the target. Let f_1 be the posterior target location density given failure to detect the target. Thus

$$f_1(x) = \frac{f(x)\ [(1-b(x, q_1(x))]}{1-E(q_1)}. \tag{2.7}$$

For $x\epsilon R$, let $b_1(x, \cdot)$ be the conditional local effectiveness function at x given that $q_1(x)$ search effort density was placed at x and the target not detected. Then

$$b_1(x, y) = \frac{b(x, q_1(x)+y) - b(x, q_1(x))}{1-b(x, q_1(x))}. \tag{2.8}$$

Suppose that h is an allocation of effort which is added onto the original allocation q_1, so that the resulting total effort density is $q_1(x)+h(x)$ for $x\epsilon R$. Then

$$E_1(h) = \int_R f_1(x)\, b_1(x, h(x))\, dx \tag{2.9}$$

is the conditional probability of detecting the target given that allocation q_1 failed.

Fix an increment of effort T. Suppose h^* has the property that $\int_R h^*(x)\, dx = T$ and

$$E_1(h^*) = \max\left\{E_1(h) : h \geqslant 0 \text{ and} \int_R h(x)dx = T\right\}.$$

Then h^* is sometimes called a *conditionally optimal* search. If we let $q_2 = q_1 + h^*$, then we claim (q_1, q_2) is an incrementally optimal sequence. To see this, we observe that by (2.7) and (2.8),

$$E_1(h) = \frac{E(q_1 + h) - E(q_1)}{1 - E(q_1)}.$$

Thus maximizing E_1 subjected to $h \geqslant 0$ and $\int_R h(x)\, dx = T$ is equivalent to maximizing E subject to $q \geqslant q_1$ and $C(q) = C(q_1) + T$. The claim now follows from the definition of incremental optimality, and we see that the concepts of incremental and conditional optimality coincide for searches of the type discussed in this paper. Hence, under the conditions of Theorem 2.1 or 2,2, a sequence of conditionally optimal searches $(h_1, h_2, \ldots)$ produce, by setting $q_i = \sum_{k=1}^{i} h_k$ a totally optimal sequence $(q_1, q_2, \ldots)$ of search allocations.

3. EXISTENCE THEOREM

In this section we find conditions under which uniformly optimal search plans or allocation schedules exist. More precisely let J be an interval of real numbers. Then an *allocation schedule* over J is a Borel function η defined on $R \times J$ such that for $T \epsilon J$, $\eta(\cdot, T) \epsilon \Phi$ and for *a.e.* $x \epsilon R$, $\eta(x, \cdot)$ is increasing. An allocation schedule η is *uniformly optimal* if

$$(3.1) \qquad C(\eta(\cdot, T)) = T \text{ and } E(\eta(\cdot, T)) = \max\{E(q) : C(q) \leqslant T\} \qquad \text{for } T \epsilon J.$$

This definition is a generalization of the definition of uniform optimality for search plans given by Arkin in [1]. In the special case where $E(q)$ gives the probability of detection resulting from the allocation of search effort, q, we call η a *search plan*. Then a uniformly optimal search plan maximizes the probability of detection at each instant during the search. In order to prove the existence of such allocation plans we define a notion of coverability similar to the one in [7].

Suppose p is a real-valued function defined on an interval J of real numbers. If p is concave, then throughout the interior of its domain, p' exists a.e. and is decreasing. Moreover, if p is continuous, then $p(t) - p(s) = \int_s^t p'(r)\,dr$ for s, $t \epsilon J$. By an *extreme* point of a concave function p, we mean a point on its graph which does not lie on a chord joining two other points on the graph.

Define $m(x, \cdot)$ to be the minimal concave majorant of $e(x, \cdot)$ for all $x \epsilon R$ for which such a majorant exists. We say that m covers e if the following conditions are satisfied.

(i) For a.e. $x \epsilon R$, $m(x, \cdot)$ exists and is continuous.

(ii) m is a Borel function.

(iii) $e(x, y) = m(x, y)$ whenever $(y, m(x, y))$ is an extreme point of $m(x, \cdot)$.

Note that condition (iii) is equivalent to assuming that $e(x, \cdot)$ is upper semi-continuous at y such that $(y, m(x, y))$ is an extreme point of $m(x, \cdot)$. For $q \epsilon \Phi$ we define

$$M(q) = \int_R m(x, q(x))\,dx$$

whenever the integral on the right exists.

Differentiation is always with respect to the last component of the argument, and is denoted by a prime, e.g., for $(x, y)\epsilon\Omega$, $e'(x, y) = \lim_{h\to 0} [e(x, y+h) - e(x, y)]/h$. Let m cover e. If a function $q\epsilon\Phi$ and a real number λ satisfy, for a. e. $x\epsilon R$,

$$m'(x, y) \geq \lambda \qquad \text{for a. e. } y \text{ such that } L(x) < y < q(x)$$

$$m'(x, y) \leq \lambda \qquad \text{for a. e. } y \text{ such that } q(x) < y < U(x),$$

then we say that the pair (q, λ) satisfies the Neyman-Pearson inequalities. When $e(x,\cdot)$ and $m(x,\cdot)$ are increasing and $U(x) = \infty$, it is convenient to define

$$e(x, \infty) = \lim_{y\to\infty} e(x, y) \text{ and } m(x, \infty) = \lim_{y\to\infty} m(x, \infty).$$

Before proceeding with our main existence result, we prove two lemmas which will be useful in this section. Lemma 3.1 relates closely to Theorem 1 and Remark 3 of [5].

LEMMA 3.1: Let m cover e. If there is a $q^*\epsilon\Phi$ such that $E(q^*) > -\infty$ and $\lambda \geq 0$ such that for a. e. $x\epsilon R$

$$\text{(3.2)} \qquad \begin{cases} \text{(i)}\ \ m'(x, y) \geq \lambda & \text{for a. e. } y \text{ such that } L(x) < y < q^*(x) \\ \text{(ii)}\ \ m'(x, y) \leq \lambda & \text{for a. e. } y \text{ such that } q^*(x) < y < U(x) \\ \text{(iii)}\ \ e(x, q^*(x)) = m(x, q^*(x)), & \end{cases}$$

then

$$\text{(3.3)} \qquad E(q^*) = \max\{E(q) : C(q) \leq C(q^*)\}.$$

PROOF: By (3.2) (iii), $M(q^*)$ exists. It is an easily shown Neyman-Pearson result (see Theorem 1 of [5]) that for $\lambda \geq 0$, (i) and (ii) imply

$$\text{(3.4)} \qquad M(q^*) = \max\{M(q) : C(q) \leq C(q^*)\}.$$

Suppose that there is an $r\epsilon\Phi$ such that $E(r) > E(q^*)$ and $C(r) \leq C(q^*)$. Since m majorizes e, we have

$$M(r) \geq E(r) > E(q^*) = M(q^*),$$

which contradicts (3.4). This proves the lemma.

For $\lambda > 0$ and x such that $m(x,\cdot)$ exists, we define

$$\varphi_u(x, \lambda) = \sup\{y : y = L(x) \text{ or } m'(x, y) \geq \lambda\}$$

$$\varphi_\ell(x, \lambda) = \inf\{y : y = U(x) \text{ or } m'(x, y) \leq \lambda\}.$$

Then for $\lambda > 0$, we let

$$I_\ell(\lambda) = \int_R \varphi_\ell(x, \lambda)\,dx, \qquad I_u(\lambda) = \int_R \varphi_u(x, \lambda)\,dx.$$

The functions φ_ℓ and φ_u will be our main tools for constructing solutions to the constrained extremal problems considered here. The following lemma displays some of the properties of these functions.

LEMMA 3.2: Suppose m covers e and for a.e. $x\epsilon R$, $e(x,\cdot)$ is increasing. If $-\infty < E(L) \leqslant E(U) < \infty$ and $|C(L)| < \infty$, then the following hold:

(a) $\varphi_u(\cdot, \lambda)\epsilon\Phi$ and $\varphi_\ell(\cdot, \lambda)\epsilon\Phi$ for $\lambda > 0$.

(b) I_ℓ and I_u are finite and decreasing.

(c) $\varphi_\ell(x, \cdot)$ and I_ℓ are right continuous and $\varphi_u(x, \cdot)$ and I_u are left continuous.

(d) $(\varphi_\ell(\cdot, \lambda), \lambda)$ and $(\varphi_u(\cdot, \lambda), \lambda)$ satisfy the Neyman-Pearson conditions.

(e) A pair (q, λ), where $q\epsilon\Phi$ and $\lambda \geqslant 0$, satisfies the Neyman-Pearson inequalities if, and only if,

$$\varphi_\ell(x, \lambda) \leqslant q(x) \leqslant \varphi_u(x, \lambda) \qquad \text{for a.e. } x\epsilon R.$$

(f) For any $\lambda > 0$, we may find a Borel function α defined on $R \times [I_\ell(\lambda), I_u(\lambda)]$, such that

(1) $\alpha(x, \cdot)$ is increasing for a.e. $x\epsilon R$,

(2) $C(\alpha(\cdot, T)) = T$ for $I_\ell(\lambda) \leqslant T \leqslant I_u(\lambda)$,

(3) $(\alpha(\cdot, T), \lambda)$ satisfies the Neyman-Pearson inequalities for all $I_\ell(\lambda) \leqslant T \leqslant I_u(\lambda)$.

(g) $\lim_{\lambda\to\infty} I_u(\lambda) = C(L)$.

(h) For $\lambda > 0$ and x such that $m(x, \cdot)$ exists, $(\varphi_u(x, \lambda), m(x, \varphi_u(x, \lambda)))$, and $(\varphi_\ell(x,\lambda), m(x, \varphi_\ell(x,\lambda)))$ are extreme points of $m(x, \cdot)$.

PROOF: A straightforward verification shows that $\varphi_\ell(\cdot, \lambda)$ and $\varphi_u(\cdot, \lambda)$ are Borel functions for each $\lambda > 0$ and that (a) holds. Thus, the integrals, $I_\ell(\lambda)$ and $I_u(\lambda)$ are well defined for each $\lambda > 0$.

For a.e. $x\epsilon R$, the following hold. Since $e(x, \cdot)$ is increasing, $m(x, \cdot)$ is increasing. If $U(x)$ is finite, then $(U(x), m(x, U(x)))$ is an extreme point and $m(x, U(x)) = e(x, U(x))$. If $U(x) = \infty$, then the increasing nature of $e(x, \cdot)$ and the minimal nature of $m(x, \cdot)$ yields $m(x, \infty) = e(x, \infty)$. Since $|C(L)| < \infty$, $L(x)$ is finite and $m(x, L(x)) = e(x, L(x)\cdot)$.

To prove (b), we observe that

$$-\infty < E(L) = M(L) \leqslant E(U) = M(U) < \infty.$$

Thus, for a.e. $x\epsilon R$, $m(x, L(x))$ and $m(x, U(x))$ are finite.

Since $m(x, \cdot)$ is increasing, we have for a. e. $x\epsilon R$,

$$m(x, U(x)) - m(x, L(x)) \geqslant \int_{L(x)}^{z} m'(x, y)dy \geqslant (z - L(x))\, m'(x, z) \qquad \text{for } L(x) < z < U(x).$$

Thus, $m'(x, z) \leqslant (m(x, U(x)) - m(x, L(x)))/(z - L(x))$, and it follows that

$$\varphi_u(x, \lambda) \leqslant \frac{1}{\lambda}[m(x, U(x)) - m(x, L(x))] + L(x) \qquad \text{for } \lambda > 0.$$

Hence,

$$-\infty < C(L) \leqslant I_\ell(\lambda) \leqslant I_u(\lambda) \leqslant \frac{1}{\lambda}[M(U) - M(L)] + C(L) < \infty \qquad \text{for } \lambda > 0$$

which proves that I_ℓ and I_u are finite. The decreasing nature of $m'(x, \cdot)$ for a. e. $x \epsilon R$ guarantees that $\varphi_u(x, \cdot)$ and $\varphi_\ell(x, \cdot)$ are decreasing for a. e. $x \epsilon R$. Thus, (b) follows.

The left continuity of $\varphi_u(x, \cdot)$ and the right continuity of $\varphi_\ell(x, \cdot)$ for a. e. $x \epsilon R$ follow from their definitions and the decreasing nature of $m'(x, \cdot)$. The monotone convergence theorem and the finiteness of I_u and I_ℓ may be used to show the left and right continuity of I_u and I_ℓ, respectively. Thus, (c) holds.

Properties (d) and (e) follow directly from the definition of φ_ℓ and φ_u. In order to prove (f), we use a device of Arkin's [1] and define for $0 \leq s \leq \infty$

$$h_\lambda(x, s) = \begin{cases} \varphi_u(x, \lambda) & \text{if } |x| < s \\ \varphi_\ell(x, \lambda) & \text{if } |x| \geq s \end{cases}$$

and

$$H_\lambda(s) = \int_R h_\lambda(x, s)dx.$$

By the monotone convergence theorem, H_λ is continuous. Moreover, H_λ is increasing and

$$H_\lambda(0) = I_\ell(\lambda), \qquad H_\lambda(\infty) = I_u(\lambda).$$

Thus for $I_\ell(\lambda) \leq T \leq I_u(\lambda)$, we may choose $\xi(T)$ such that $H_\lambda(\xi(T)) = T$. Defining $\alpha(x, T) = h_\lambda(x, \xi(T))$ for $x \epsilon R$, we see that α satisfies conditions (1) and (2) of (f). Condition (3) follows from (e). Property (g) follows easily from the monotone convergence theorem and the definition of φ_u. Property (h) may be verified from the definitions of φ_ℓ, φ_u, and an extreme point. This completes the proof.

THEOREM 3.3: Suppose m covers e, and for a. e. $x \epsilon R$, $e(x, \cdot)$ is increasing. If $-\infty < E(L) \leq E(U) < \infty$ and $|C(L)| < \infty$, then there exists a uniformly optimal allocation schedule η over $[C(L), C(U))$.

PROOF: We consider first the case where $I_u(0) \equiv \lim_{\lambda \to 0} I_u(\lambda) = C(U)$. The case $I_u(0) < C(U)$ requires only routine modifications which are discussed at the end of the proof. We take $\eta(\cdot, C(L)) = L$.

Since I_u is monotone, it has only a countable number of discontinuities. Let K be a countable index set such that $\{\lambda_k : k \epsilon K\}$ is the set of discontinuity points of I_u. Let $J_k = [I_\ell(\lambda_k), I_u(\lambda_k)]$ for $k \epsilon K$. The intervals J_k are disjoint and are the jump intervals at the discontinuity points of I_u. For $T \epsilon (C(L), C(U)) - \bigcup_{k \epsilon K} J_k$, let

$$\lambda^*(T) = \sup\{\lambda : I_u(\lambda) = T\}.$$

By the left continuity of I_u, $I_u(\lambda^*(T)) = T$.

For $T \epsilon J_k$, let $\lambda^*(T) = \lambda_k$ and choose a function α_k defined on $R \times J_k$ to have the properties of α in (f) of Lemma 3.2. Define

$$\eta(x, T) = \begin{cases} \varphi_u(x, \lambda^*(T)) & \text{if } C(L) < T < C(U) \text{ and } T \notin \bigcup_{k \epsilon K} J_k \\ \alpha_k(x, T) & \text{if } T \epsilon J_k \text{ and } k \epsilon K. \end{cases}$$

Then for each $C(L) < T < C(U)$, $C(\eta(\cdot, T)) = T$ and $(\eta(\cdot, T), \lambda^*(T))$ satisfies the Neyman-Pearson conditions. Since m covers e and property (h) of Lemma 3.2 holds, we have that for each $C(L) < T < C(U)$, $e(x, \eta(x, T)) = m(x, \eta(x, T))$ for a.e. $x \epsilon R$. Thus, by Lemma 3.1, η satisfies 3.1.

To verify that $\eta(x, \cdot)$ is increasing for a.e. $x \epsilon R$, we let R' be the set of $x \epsilon R$ such that $m(x, \cdot)$ exists. Then by the fact that m covers e, $R - R'$ has measure 0. Suppose it is not the case that $\eta(x, \cdot)$ is increasing for a.e. $x \epsilon R$. Then there is an $x \epsilon R'$ and numbers S and T, such that $C(L) < T < S < C(U)$ and

$$\eta(x, S) < \eta(x, T). \tag{3.4}$$

Since $(\eta(\cdot, T), \lambda^*(T))$ and $(\eta(\cdot, S), \lambda^*(T))$ satisfy the Neyman-Pearson inequalities for all $x \epsilon R'$, we have

$$\lambda^*(T) \leqslant m'(x, y) \leqslant \lambda^*(S) \qquad \text{for a.e. } y \text{ such that } \eta(x, S) < y < \eta(x, T).$$

One may check that λ^* is a decreasing function, so that $\lambda^*(T) = \lambda^*(S)$. Thus, for some $k \epsilon K$, T and S are both in the closure of J_k. However, $\eta(x, \cdot)$ is constructed by property (f) of Lemma 3.2 to be increasing on the closure of J_k. This contradicts (3.4) and proves the theorem for the case where $I_u(0) = C(U)$.

If $I_u(0) < C(U)$, we proceed as before for $C(L) < T \leqslant I_u(0)$. We then define

$$\varphi_u(x, 0) = \lim_{\lambda \to 0} \varphi_u(x, \lambda).$$

From the increasing nature of $e(x, \cdot)$, it follows that if $q \epsilon \Phi$ and if $q(x) \geqslant \varphi_u(x, 0)$ for $x \epsilon R$, then q will satisfy (3.2) with $\lambda = 0$. Hence, to complete the definition of $\eta(x, \cdot)$ for $I_u(0) < T < C(U)$, one need only choose η so that $\eta(x, T) \geqslant \varphi_u(x, 0)$ and $C(\eta(\cdot, T)) = T$ which may be easily done. This completes the proof.

Observe that the hypotheses of Theorem 3.3 may be weakened to require that $m(x, \cdot)$ rather than $e(x, \cdot)$ be increasing for a.e. $x \epsilon R$. The theorem remains unchanged except that η must be restricted to $[C(L), I_u(0)]$. This is no real restriction since for $q \geqslant \varphi_u(\cdot, 0)$, $E(q) \leqslant E(\varphi_u(\cdot, 0))$.

Theorem 2 of Arkin [1] is similar to Theorem 3.3 above with the exception that [1] claims that there exists a function β such that

$$\eta(x, T) = \int_0^T \beta(x, s) ds, \qquad C(\eta(\cdot, T)) = T,$$

and η is uniformly optimal. However, the following is a counterexample to Theorem 2 of [1]. (Moreover, the proof in [1] is not sufficient to show the truth of Theorem 3.3.)

Let $R = [0, 1]$, $L = 0$, $U = \infty$, and

$$e(x, y) = \begin{cases} 0, & 0 \leqslant y < 1 \\ 1, & y \geqslant 1 \end{cases}$$

for $x \epsilon R$. It is clear that any uniformly optimal search plan η must have the property that for a.e. $x \epsilon R$, $\eta(x, \cdot)$ jumps from 0 to 1 at some point T, but there is no function β which produces this behavior for η.

Under the conditions of Theorem 3.3. we have shown that there exists, for any $C(L) \leqslant T < C(U)$, a q^* such that $C(q^*) = T$ and $E(q^*) = \max\{E(q) : q \epsilon \Phi \text{ and } C(q) \leqslant T\}$. Theorem 8 of [7] provides a similar existence result whenever m covers e and $-\infty < C(L) \leqslant C(U) < \infty$. In comparison, Theorem 3.3 of this paper removes the restriction that $C(U) < \infty$, but adds monotonicity conditions on $e(x, \cdot)$ and boundedness conditions on E. In [6] there is also a discussion of related existence theorems.

One might conjecture that Theorem 3.3 would remain true without assuming that $e(x, \cdot)$ is increasing, provided that we assumed $|E(q)| < B$ for some number B and all $q\epsilon\Phi$. Similarly, one might conjecture that the restriction $C(L) > -\infty$ could be omitted. However, the following two counterexamples show both of these conjectures to be false.

EXAMPLE 3.4: Let $R = [1, \infty)$, $L = 0$, and $U(x) = x + 1/x^2$ for $x \epsilon R$.
For $x\epsilon R$, define

$$e(x, y) = \begin{cases} y, & 0 \leqslant y \leqslant 1/x^2 \\ \dfrac{1}{x^2} - \dfrac{y - 1/x^2}{x^3}, & 1/x^2 < y \leqslant x + 1/x^2. \end{cases}$$

Note that $m = e$ and that $|E(q)| \leqslant 1$ for all $q\epsilon\Phi$. Suppose q^* is optimal and

$$\infty > C(q^*) > 1.$$

By Corollary 7.2 of [6] there exists a λ such that

$$e(x, q^*(x)) - \lambda q^*(x) = \sup\{e(x, y) - \lambda y : 0 \leqslant y < x + 1/x^2\} \qquad \text{for a.e. } x\epsilon R. \tag{3.5}$$

Since $e(x, \cdot)$ is concave for $x\epsilon R$, this implies

$$e'(x, y) \geqslant \lambda \qquad \text{for } 0 < y < q^*(x)$$

$$e'(x, y) \leqslant \lambda \qquad \text{for } q^*(x) < y < x + 1/x^2, \qquad \text{for a.e. } x\epsilon R.$$

One may check that if $\lambda \geqslant 0$, $C(q^*) \leqslant 1$. Thus, the above λ must be negative. It follows from the above inequalities that $q^*(x) = x + 1/x^2$ for $x^3 > -1/\lambda$. Thus, $C(q^*) = \infty$ which contradicts our assumption that $C(q^*) < \infty$. Thus, one cannot replace the monotonicity of $e(x, \cdot)$ by boundedness of E in Theorem 3.3.

EXAMPLE 3.5: Let $R = [1, \infty)$, $L = -1$, $U = 1$, and

$$e(x, y) = y/x^2 \qquad \text{for } -1 \leqslant y \leqslant 1.$$

Observe that $e = m$ and all the conditions of Theorem 3.3 are satisfied except that $C(L) = -\infty$. Suppose q^* is optimal and $C(q^*)$ is finite. Again by Corollary 5.2 of [6] there exists a λ such that

$$e(x, q^*(x)) - \lambda q^*(x) = \sup\{e(x, y) - \lambda y : -1 \leqslant y \leqslant 1\} \qquad \text{for a.e. } x\epsilon R.$$

Hence,

$$e'(x, y) \geqslant \lambda, \qquad -1 < y < q^*(x)$$

$$e'(x, y) \leqslant \lambda, \qquad q^*(x) < y < 1, \qquad \text{for a.e. } x\epsilon R.$$

It follows that

$$q^*(x) = \begin{cases} 1 & \text{for } x^2 < 1/\lambda \\ -1 & \text{for } x^2 > 1/\lambda. \end{cases}$$

Hence, either $C(q^*) = -\infty$ or $C(q^*) = +\infty$ contrary to the assumption that $C(q)$ is finite. Thus, we cannot omit the condition $C(L) > -\infty$ in Theorem 3.3.

REMARK 3.6: In the search theory case where $L(x) = 0$, $U(x) = \infty$ for $x\epsilon R$ and $e(x, y) = f(x) b(x, y)$ for $(x, y)\epsilon\Omega$, the conditions of Theorem 3.3 will be satisfied if $b(x, \cdot)$ is right continuous for $x\epsilon R$. Since $b(x, \cdot)$ is increasing and $|E(q)| \leqslant 1$, for $q\epsilon\Phi$, the only condition that is not obviously satisfied is the coverability condition. However, since $b(x, \cdot)$ is increasing and right-continuous, it is upper semi-continuous. Thus, $e(x, \cdot)$ has a minimal concave majorant $m(x, \cdot)$ which is continuous, and one may check that $e(x, y) = m(x, y)$ whenever $(y, m(x, y))$ is an extreme point of $m(x, \cdot)$. It can be shown that since e is Borel, m is a.e. equal to a Borel function. Thus the conditions for coverability are satisfied. It follows that whenever the local effectiveness function is right continuous and the target location distribution is given by a density function on Euclidean n-space, a uniformly optimal search plan exists. Note that uniformly optimal search plans may be used to produce sequences which are both incrementally and totally optimal.

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