JMM 2026 – Metron's Daily Math Problems – Sheet B (Monday, January 5)

B1) Train riders trickle in, and one escalator is sufficient to handle the trickle. But when a train arrives, you need two escalators going out to handle the surge. Source: *Car Talk* puzzler, 9/20/2004: Curious Escalator Math.

This isn't a math problem *per se*, but a cautionary tale in mathematical modeling. It is tempting to impose convenient assumptions on real-world problems to make them tractable: e.g., independent passenger arrival times. And, in fact, independent, uniformly distributed *arrival* times is a reasonable assumption ("train riders trickle in"), but independent *departure* times is not. Thus it is important to understand the salient features of a problem before mathematical modeling begins.

B2) Reg's expected wait time is 5 minutes: the expectation of a random variable uniformly distributed on [0,10]. Randy's expected wait time is 10 minutes. Because the exponential distribution is memoryless, at any given time the expected wait until the next arrival is 10 minutes, as is the expected time elapsed since the previous arrival. Although the expected time between buses is 10 minutes in both Reg's case and Randy's, Randy will need to wait twice as long, on average.

This waiting-time paradox occurs because an observer arriving at a (uniformly distributed) random time is more likely to end up in a long interarrival interval than a short one. If m is the mean interarrival time of buses and w is the mean wait time of an observer, then the variance of interarrival times is $m\,(2w - m)$. Thus, for a given mean interarrival time m, the mean wait time w increases with the variance of interarrival times. Reg's zero-variance buses yield the minimal possible wait time $w = m/2$.

B3) 36/95. This was the final problem of a math contest. Only 2% of participants answered it correctly. The problem is straightforward when viewed as an M/G/n/n queue, but this is a rather specialized field. George Lowther provides a first-principles answer to an equivalent problem. Here is a solution that leverages notions from queueing theory.

Let the time a customer spends eating their ice cream be 1, and let λ denote the customer arrival rate in general (with $\lambda = 2$ in this case). It is simpler to think instead of "Hilbert's Ice Cream Shop" with an infinite sequence of stools 1, 2, 3, …, where customers take the first unoccupied stool. The occupation probability of the middle stool in the original problem is the same as that of stool 3 in Hilbert's shop.

Assuming stationarity, the number of occupied stools is Poisson distributed with mean λ. I.e., the probability of exactly i stools being occupied is $\pi_i = e^{-\lambda}\lambda^i / i!$. The probability $\pi_{i,n}$ of exactly i of the first n stools being occupied is proportional to π_i:

$$\pi_{i,n} = \frac{\lambda^i / i!}{\Lambda_n}, \quad \text{where} \quad \Lambda_n \doteq \sum_{i=0}^{n} \frac{\lambda^i}{i!}.$$

Setting $i = n$ yields the Erlang B formula for $\pi_{n,n}$, the probability that all of the first *n* stools are occupied. The occupation probability p_3 of stool 3 follows from Little's law and the PASTA property: p_3 equals the rate at which new customers sit on stool 3 times the average time they spend there. This equals λ (the overall arrival rate) × ($\pi_{2,2} - \pi_{3,3}$) (the probability that stools 1 and 2 are occupied but stool 3 is not) × 1 (the mean time a customer spends in the stool). Therefore the requested occupation probability is

$$p_3 = \lambda\left(\frac{\lambda^2/2}{1+\lambda+\lambda^2/2} - \frac{\lambda^3/6}{1+\lambda+\lambda^2/2+\lambda^3/6}\right) = 2\left(\frac{2}{1+2+2} - \frac{4/3}{1+2+2+4/3}\right) = \frac{36}{95}.$$