JMM 2026 – Metron's Daily Math Problems – Sheet C (Tuesday, January 6)

C1) This problem is begging for the pigeonhole principle to be applied to it. If instead the problem had asked us to show two consecutive integers can be chosen, we could define the n "pigeonholes" $\{1,2\}$, $\{3,4\}$, …, $\{2n-1,2n\}$, knowing that two of the $n+1$ "pigeons" must occupy the same hole. But in this case, how do we define our pigeonholes?

The temptation is to look at the "$2n$" and think the n pigeonholes we define are meant to have size 2. Instead, the Book Proof defines the pigeonholes to be equivalence classes of the relation "having the same odd part." The n pigeonholes are $\{1,2,4,8,\ldots\}$, $\{3,6,12,\ldots\}$, $\{5,10,20,\ldots\}$, …, $\{2n-1\}$. Any set of $n+1$ distinct positive integers chosen from $\{1, 2, 3, \ldots, 2n\}$ must contain two integers from the same pigeonhole. The ratio of the larger to the smaller of these is an integer (a power of 2, in fact).

C2) The generating function of the arithmetic sequence $(a, a+d, a+2d, \ldots)$ is $z^a/(1-z^d)$, which converges absolutely for $|z| < 1$. Were the partition possible, then $z/(1-z)$ could be expressed as a sum of terms of the form $z^a/(1-z^d)$, each with a different stride d. Let D be the largest of these d. Taking the limit as $z \to e^{2\pi i/D}$ yields an infinite result for exactly one term in the sum. Therefore, the sum does not have a finite limit as $z \to e^{2\pi i/D}$, whereas $z/(1-z)$ does (because requiring at least two different d implies $D > 1$), which is a contradiction. Cf. this MathStackExchange post, which cites D. J. Newman as the source of this argument.

C3) In Hilbert's prison, the inmates are labeled by a bijection with the positive integers. In the morning, they will line up in order and each be assigned a red or a blue hat. Each will face right, seeing all hats of higher-numbered prisoners but no others. Starting with prisoner 1, each prisoner will guess the color of their own hat. Guessing wrong kills the prisoner.

The prisoners can meet beforehand to strategize. They are allowed to invoke the axiom of choice in formulating their strategy. How can they ensure that only a finite number of them die, even if they *cannot hear* the other prisoners' responses? How can they ensure that at most one dies if they *can hear* the others' responses?

Define two sequences of hat colors to be equivalent if they differ at only finitely many locations. Now consider the equivalence classes of this relation. In their planning meeting, the prisoners invoke the axiom of choice to select a canonical representative for each equivalence class. In the morning, although each prisoner can only see the hats to their right, this is enough information to determine the equivalence class of the full sequence of hat colors. The prisoners then use the canonical representative of this class as their guess for the full sequence of colors: in particular, for their own hat's color. Although each prisoner has only a 50% chance of guessing their own hat color correctly, the canonical representative they're using has only a finite number of bad entries. Only those unlucky prisoners die. Cf. Wikipedia for further discussion.

If the prisoners can hear previous responses, then, once they've chosen their canonical representative sequence, the prisoners can use the same strategy that solves the popular 100-prisoner version.